THE

ELEMENTARY PRINCIPLES

OF

MECHANICS.

VOL. II.

STATICS.

BY

A. JAY DU BOIS, C.E., PH.D.,

Professor of Civil Engineering
in the Sheffield Scientific School of Yale University;
Author of "Elements of Graphical Statics";
"The Strains in Framed Structures";
etc.

FIRST EDITION.

FIRST THOUSAND.

NEW YORK:
JOHN WILEY & SONS,
53 EAST TENTH STREET.
1894.

ROBERT DRUMMOND, ELECTROTYPER AND PRINTER, NEW YORK.

NOTE.

The large type by itself constitutes an abridged course. Articles in small type are for advanced students. Articles containing applications of the Calculus are inclosed in brackets.

MECHANICS.

TABLE OF CONTENTS.

Vol. II.

STATICS.

INTRODUCTION.

CHAPTER I.

MATTER, BODY, PARTICLE, INERTIA. FORCE. DYNAMICS.

CHAPTER II.

DENSITY. SPECIFIC MASS. DETERMINATION OF SPECIFIC MASS.

CHAPTER III.

CENTRE OF MASS.

CHAPTER IV.

COMPOSITION AND RESOLUTION OF FORCES. ATTRACTION.

STATICS.

CHAPTER I.

CONCURRING FORCES.

CHAPTER II.

PARALLEL FORCES.

CHAPTER III.

NON-CONCURRING FORCES IN GENERAL.

CHAPTER IV.

NON-CONCURRING CO-PLANAR FORCES.

CHAPTER V.

EQUILIBRIUM OF STRINGS.

CHAPTER VI.

GRAPHICAL STATICS. CO-PLANAR FORCES.

CHAPTER VII.

WORK.

CHAPTER VIII.

CONSTRAINED EQUILIBRIUM. SMOOTH CURVE OR SURFACE.

CHAPTER IX.

CONSTRAINED EQUILIBRIUM. ROUGH CURVE OR SURFACE.

APPLICATIONS OF STATICS.

CHAPTER I.

RETAINING WALLS. DAMS. EARTH SLOPES.

CHAPTER II.

STRENGTH AND ELASTICITY OF MATERIALS.

CHAPTER III.

THEORY OF FLEXURE.

CHAPTER IV.

THE CONTINUOUS GIRDER.

DYNAMICS.

INTRODUCTION.

CHAPTER I.

MATTER, BODY, PARTICLE. INERTIA. FORCE. DYNAMICS.

FORCE PROPORTIONAL TO ACCELERATION. UNIFORM AND VARIABLE FORCE. MASS. UNIT OF MASS. MEASUREMENT OF MASS. RELATION BETWEEN FORCE, MASS AND ACCELERATION. ABSOLUTE UNIT OF FORCE. GRAVITATION UNIT OF FORCE.

Matter—Body—Particle.—What matter is in itself we do not know. We recognize it as existing in space and possessing certain observed properties, such as extension and impenetrability.

Any limited portion of matter we call a **body**. A body so small that, so far as its motion is concerned, we can disregard its size we call a **material point or particle**. Just as a mathematical point, having no dimensions, cannot rotate, but can have motion of translation only, so a material point or particle is considered as having motion of translation only.

Every body may be considered as a system composed of such material points or particles.

The diagram representation of a particle is then a mathematical point, having position only.

When a body has motion of translation only, the motion of every one of its points at any instant is the same (page 13, Vol. I), and in such case we may then consider the entire body, *whatever its size*, as a particle and represent it by a mathematical point.

Hence, whatever the size of a body, *when we consider its motion of translation only*, we may treat the body as a particle and represent it by a point.

Inertia—Force.—It is a fact of universal experience that no material particle is able of itself to change its own motion. If it is at

rest, it must always remain at rest, unless acted upon by some other particle. If it is moving at any instant in a given direction with a given speed, it must always preserve that direction and speed unchanged, unless acted upon by some other particle.

We express this fact by saying that matter is *inert*, that is, has no power of itself to change its own state of rest or motion. This property of matter we call **inertia**. We recognize, then, not only extension and impenetrability, but also inertia, as properties of matter.

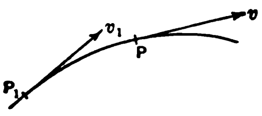
Whenever, then, the motion of a particle is observed to change either in speed or direction, we can always refer such change to the influence of some other particle upon it.

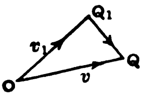
This external influence which we thus recognize as the cause of the change of motion we call **force**. We can define force, then, as the cause of change of motion of matter. We measure force, therefore, by its observed effect, that is, by the change of motion it causes.

It should be noted that inertia, as already defined, is a property of matter. To speak then, as is sometimes done, of the "force of inertia," as though *inertness* could cause change of motion or change of anything, is as unmeaning as though we should speak of "force of hardness" or "force of softness." Incapacity of self-change of motion, or inertia, cannot be spoken of as the cause of observed change. By reason of such incapacity force is necessary for change of motion.

Dynamics.—We have treated in the first portion of this work of the science of **Kinematics** (κινημα, motion), or the measurable relations of space and time, that is, of pure motion. We have therefore considered the motion of a point, or of a system of points, without reference to matter or force. But we have to deal in nature with *force* and material points or *bodies*. The science which treats of those measurable relations of matter, space and time involved in the study of the change of motion of bodies due to force is called **Dynamics** (δύναμις, force).

Force Proportional to Acceleration.—Let v_1 be the initial velocity of a material point or particle P_1 moving in any path P_1P, and v its final velocity at the end of any time t.





If we draw OQ_1 parallel and equal to v_1 and OQ parallel and equal to v, then, as we have seen, page 48, Vol. I, Q_1Q gives the integral acceleration both in direction and magnitude. Also $\frac{Q_1Q}{t}$ gives the mean acceleration or mean time-rate of change of velocity in the time t.

The limiting magnitude and direction of $\frac{Q_1Q}{t}$ when the time t is indefinitely small is the acceleration, or instantaneous time-rate of change of velocity.

Now this change of velocity is due to the force at that instant. If there were no force, v_1 would remain unchanged both in magnitude and direction.

Since we can only measure force by its effects, and since here the effect is shown by change of velocity, the force must be proportional to this change of velocity.

We conclude, therefore, that the *direction of the force is the same as the direction of the acceleration it causes, and the magnitude of the force is proportional to the magnitude of the acceleration it causes.*

Mechanical Illustration of Force.—The student may figure to himself such a force as the pressure or pull of an imponderable spiral spring acting upon the body, the axis of the spring having always the direction of the acceleration, and the spring moving with the body so that its pressure or pull is exerted during the entire time of action and is alway proportional to the acceleration.

If the acceleration changes in direction, the axis of the spring changes, so that it always has the same direction as the acceleration.

If the acceleration changes in magnitude, the pull or push of the spring changes accordingly.

If the acceleration is uniform, that is, does not change either in direction or magnitude, the axis of the spring does not change in direction and its pull or push is constant.

The force of gravity upon bodies near the surface of the earth is like the action of such a spring. Its action is practically constant in intensity and direction.

The student should note that the direction of the force or acceleration is not necessarily that of the motion, except in the case of rectilinear motion.

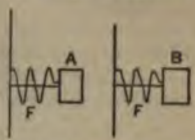
Thus in the case of a point moving with uniform speed in a circle, the direction of motion at any instant is tangent to the circle, but the acceleration is always directed towards the centre (page 53, Vol. I).

In the case of a projectile, the motion at any instant is tangent to the path, but the acceleration is always vertical and downwards.

Uniform and Variable Force.—A force, then, like acceleration, page 49, Vol. I, is uniform or constant when it has the same magnitude and the same direction whatever the time of action. When either the magnitude or direction changes it is variable.

Criterion of the Action of a Force.—The action of a force on a particle, then, is made evident by the change of motion it causes. If the particle is at rest or moves with uniform speed in a straight line, there is no force acting upon it. If either the speed changes or the direction of motion changes, a force must act upon it to cause such change. The magnitude of the acceleration is proportional to the magnitude of the force, and the direction of the acceleration is the direction of the force. The force is uniform when the acceleration is uniform, and variable when the acceleration is variable.

Mass.—Let such a spring, F, as described, act with constant pressure in a constant direction upon a given body A for a given time.



Then the acceleration or change of velocity per second is constant and in the direction of the force or axis of the spring.

Let the *same spring* act upon another body, B, with the same constant pressure in the same constant direction for the same time. Then the acceleration or change of velocity per second in this case is also constant and in the direction of the axis of the spring.

If the magnitude of the acceleration in the second case is equal to the magnitude in the first case, the body B is said to have the same **mass** as the body A. In general,

Equal masses are those to which the same uniform force gives the same acceleration in the direction of the force in the same time.

Unit of Mass.—We take as the unit of mass the standard pound avoirdupois, or the standard gram, or the standard kilogram.

These are definite bodies (page 5, Vol. I). Any other body which when acted upon by any given constant force would receive in the same time the same acceleration in the direction of the force as the standard mass under the same circumstances is an equal mass.

When, then, the mass of a body is unity, or one unit of mass, the same constant force acting upon it gives it the same acceleration in the same time in the direction of the force that the standard mass would receive under like circumstances.

Measurement of Mass.—We know by experiment that the force of gravity, or the earth's attraction at any place, gives to all bodies falling in vacuum, whatever their nature, the same acceleration in the same time.

This acceleration is vertical or in the direction of the force of gravity which causes it.

When two bodies exactly balance in an equal-armed balance, we also know that the force of gravity on each must be the same.

Since then, under the action of this equal force, each body would acquire the same acceleration in the same time in the direction of the force, their masses are equal.

By means of the balance, then, we can readily duplicate standard masses. By finding how many such standard masses balance any given body, that is, by "weighing" the body, we can determine its mass relatively to the standard.

Thus if any body exactly balances 2, 3 or 4 standard pounds or kilograms or grams, its mass is 2, 3 or 4 times the mass of the standard used.

Mass Independent of Gravity.—It must be carefully noted that the mass of a body has nothing to do with the actual intensity of the force of gravity. This varies with the locality and the height above sea-level in the same locality. But two bodies of equal mass which therefore exactly balance in one locality would balance in any locality, because the force of gravity, whatever it may be, is always the same on each wherever they are weighed.

When we speak of a mass of one pound, one gram, or one kilogram, we refer then to a *definite quantity of matter*, not to the force of gravity acting at any place upon that matter.

But when a body balances two standard pounds, we know that the force of gravity upon that body at any locality is twice as great as for one pound. The force of gravity upon any body at any locality, or the *weight* of the body, is thus *proportional to its mass*, but the mass is independent of this weight.

The term "weighing" as applied to a balance should not be allowed to mislead. "Weighing" a body in a balance always determines its *mass* and not its *weight*, or the force of gravity upon it.

Relation between Force, Mass and Acceleration.—Since the weight of a body is proportional to its mass, and since all bodies fall in vacuum with the same acceleration under the action of gravity at any locality or of their weights, it follows that to give different bodies the same acceleration in the direction of the force, the force must be proportional to the mass.

But we also know by experiment that when we give the same body different accelerations in the direction of the force, the force is proportional to the acceleration.

In general, then, any force which produces in a given body, free to move, an acceleration in its direction, must be proportional both to the mass of the body and the acceleration.

If then $[F]$ is the unit of force adopted and F the number of units of force, $[M]$ the unit of mass and m the number of units of

mass, $[f]$ the unit of acceleration and f the number of units of acceleration in the direction of the force, we must have the relation

$$F[F] = c \,.\, m[M] \times f[f], \quad \ldots \ldots \quad (1)$$

where c is a constant number.

Equation (1) expresses the fact that force must be proportional both to the mass and the acceleration given to the mass in the direction of the force.

Unit of Force.—We see from (1) that we shall always have the numeric equation

$$F = mf \quad \ldots \ldots \ldots \ldots \quad (2)$$

if we make c unity, and

$$[F] = [M] \times [f].$$

That is, equation (2) holds provided we take as our unit of force *that constant force which will give one unit of mass one unit of acceleration in the direction of the force.*

This is called "*Gauss's* absolute unit," or the absolute unit of force, because it furnishes a standard force in any system, independent of the force of gravity at different localities.

In the foot-pound-second or "F. P. S. system," then, the absolute unit of force is that constant force which will give one pound a change of velocity in the direction of the force of one foot per second in a second. This has been called by Prof. James Thompson the **poundal**. It is then the English absolute unit of force.

The French absolute unit of force is that constant force which will give one kilogram a change of velocity in the direction of the force of one meter per second in a second.

In the centimeter-gram-second or "C. G. S. system" the absolute unit of force is the constant force which will give one gram a change of velocity in the direction of the force of one centimeter per second in a second. This is called the **dyne**.

Dimensions of Unit of Force.—Let $[F]$ represent the unit of force, $[f]$ the unit of acceleration, $[M]$ the unit of mass, $[V]$ the unit of velocity, $[L]$ the unit of distance, and $[T]$ the unit of time. Then we have

$$[F] = [M] \times [f] = [M] \times \frac{[V]}{[T]} = [M] \times \frac{[L]}{[T]^2}.$$

Weight of a Body.—The student should again be cautioned to keep clearly distinguished in his mind the difference between the mass of a body and its weight. The weight of any mass is the force with which the earth attracts it, and it therefore varies with the locality. The mass is invariable at all places.

If the weight of a body is W, and its mass m units, then, since the weight produces the acceleration g, we have from (2),

$$W = mg \text{ units of force.}$$

If m is one unit of mass, W is numerically equal to g units of force, or

one pound weighs g poundals,

one gram weighs g dynes,

according to the system in use.

Since g is about 32 ft.-per-sec. per sec., the weight of one pound is about 32 poundals, or

one poundal is the weight of about half an ounce.

Strictly speaking, it is the weight of $\frac{1}{g}$ part of a pound, where g must be taken for the locality in ft.-per-sec. per sec.

In the same way, the weight of one gram is about 981 dynes, or

one dyne is the weight of about one milligram.

Strictly speaking, it is the weight of $\frac{1}{g}$ part of a gram, where g must be taken for the locality in centimeters-per-sec. per sec.

An athlete throwing a hammer of 16 pounds in New Haven and the *same hammer* in Edinburgh has a heavier hammer to throw in the latter place, by the weight of about three tenths of an ounce more. (See page 93, Vol. I.) The mass of the hammer is of course the same in both places.

Gravitation Unit of Force.—It is often convenient to express a force by comparing it with the weight of the unit of mass at the locality. The weight of the unit of mass at the place is then the *gravitation unit of force.* It is evidently not constant. Or we can express a force by comparing it with the weight of the unit of mass at some given place. The weight of the unit of mass *at this place* is then the gravitation unit of force. In this case it is constant.

When, then, we speak of a "force of ten pounds" or a "force of ten kilograms" we mean the force of gravity at a given place upon a mass of ten pounds or ten kilograms. The expression is of course incorrect, because pound and kilogram denote mass only. The expression is thus a brief and allowable locution for the phrase —"attraction of the earth for a mass of ten pounds at the place considered."

A "force of ten pounds" means, then, a force of $10g$ poundals, where g is the acceleration of gravity in ft.-per-sec. per sec. at the place considered. A "force of ten grams" means a force of $10g$ dynes, where g is the acceleration of gravity in centimeters-per-sec. per sec. at the place considered. In all cases,

Mass (*in lbs.*) × **acceleration** (*in ft.-per-sec. per sec.*) = **Force in direction of acceleration** (*in poundals*).

If we divide the force thus found by g in ft.-per-sec. per sec., we obtain the force in gravitation units.

Mass (*in grams*) × **acceleration** (*in centimeters-per-sec. per sec.*) = **Force in direction of acceleration** (*in dynes*).

If we divide the force thus found by g in centimeters-per-sec. per sec., we obtain the force in gravitation units.

Thus if a mass of 25 pounds has an acceleration in any direction of 6.4 ft.-per-sec. per sec., the force in that direction which causes this acceleration is $25 \times 6.4 = 160$ *poundals*, or 160 times the force necessary to give a mass of one pound an acceleration of 1 ft.-per-sec. in one second. If g for the locality is 32 ft.-per-sec per sec., we can speak of this as a force of $\frac{6.4}{32} \times 25$ pounds, or a "force of 5 *pounds*," meaning thereby the force of gravity upon a mass of 5 pounds at the locality in question.

Again, if a mass of 25 grams has an acceleration in any direction of 200 centimeters-per-sec. per sec., the force in that direction which causes this acceleration is $25 \times 200 = 5000$ *dynes*, or 5000 times the force necessary to give a mass of one gram an acceleration of 1 centimeter-per-sec. in one second. If g for the locality is 981 centimeters-per-sec. per sec., we can speak of this as a force of $\frac{200}{981} \times 25$ grams, or a "force of about 5 *grams*," meaning thereby the force of gravity upon a mass of 5 grams at the locality in question.

Tension—Compression—Shear.—When a force acts to separate two particles of a body in the direction of the line joining them, it is called a force of *tension*, or **tensile force**. When it acts to bring the particles together in the direction of the line joining them, it is a force of *compression*, or **compressive force**. When it acts to displace the particles in a direction at right angles to the line joining them, it is called *shear*, or **shearing force**.

Action and Reaction.—When one body or particle presses or pulls another, it is itself pressed or pulled by this other with an equal force in an opposite direction. If we speak of the force exerted by one body or particle as **action**, we can call the force exerted on it by the other **reaction**. To every action, then, there is always an equal and opposite reaction, or the mutual actions of any two bodies are always equal and oppositely directed.

Stress.—The exertion of force upon a body or particle is thus only one side of the entire phenomenon, which really consists of the simultaneous exertion of equal and opposite forces between two bodies or particles.

When we fix our attention upon one only of the bodies or particles and, disregarding the other, consider only its action upon the first, we have called this action **force**. It is that external action due to some other particle which causes change of motion of the particle considered (page 2). But when we have both bodies or particles in mind and wish to be understood as viewing this force as one of the two mutual, equal and opposite actions between two bodies or between two particles of the same body, we call it a **stress**.

When the stress is such as to make the two bodies or particles move towards one another, or to *resist tensile force*, it is called *attraction* or **tensile stress**. When it is such as to increase their distance, or to *resist compressive force*, it is called *repulsion* or **compressive stress**. When it resists shearing force it is called **shearing stress**.

In this sense, then, we always speak of the stress *in* a body or *between* two bodies or particles; the prepositions "in" or "between" indicating at once that we have to do with one of the mutual actions between two bodies or particles. Force then is always external to the body or system considered. Stress is internal to that body or system, and resists change of configuration due to force.

External Stress.—There is, however, a sense in which we speak of stress *on* a body, and thus consider it as external, which need never be confounded with that just given.

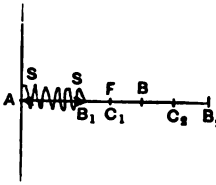
Force is often exerted upon some definite portion of the bounding surface of a body and acts then over an area. In such case the number of units in its magnitude divided by the number of units in the area gives the number of units of force *per unit of area*. When a force thus acts we may speak of it as the stress *on* the body, and the force per unit of area we call **unit stress**.

This use of stress is convenient and leads to no confusion. Where necessary to discriminate we may speak of internal stress and external stress, but in general the use of the preposition "on" and "in" or "between" sufficiently indicates the sense in which the term is used.

Strain.—The change of distance between two particles of a body in a *direction opposite to internal stress* is called **strain**.

If no internal stress exists, there is no strain, but simply **displacement**.

Illustration.—Thus let a spring whose original "unstrained" length is AB be compressed so that its length is AB_1. When we consider the external action which compresses it, we speak of the *force of compression* F. When we consider one of the mutual actions between any two points A and B_1 which resist compression, we speak of the *compressive stress* S *in the spring* at B_1 or at A.



The *strain* is the distance BB_1, or the displacement *opposite to the stress.*

If the compressive force F is removed and the spring allowed to expand to C_1, the distance B_1C_1 is not *strain* because it is not opposite in direction to the stress, but simply *displacement*. When the spring reaches B there is no stress in it. As it passes B tensile stress is developed, and any distance BC_2 is strain. The point B is the position of zero strain, and any displacement on either side of this point is strain because opposite in direction to the stress in the spring.

EXAMPLES.

(1) *With 1 ft. and 1 sec. as units of distance and time, find the unit of mass, in order that the derived unit of force may be equal to the weight of 1 lb.*

Ans. g lbs.

(2) *Find the unit of force in order that the unit of mass may be g lbs.*

Ans. g poundals.

(3) *The unit of acceleration being 6 ft.-per-sec. per sec., find (a) the unit of mass when the derived unit of force is equal to the weight of 20 lbs., and (b) the unit of force when the derived unit of mass is a mass of 20 lbs.*

Ans. (*a*) $107\frac{1}{3}$ lbs.; (*b*) 3.7 pounds weight.

(4) *The unit of mass being a mass of 10 lbs., the unit of time 1 min., and the unit of length 1 yd., compare the derived unit of force with the poundal.*

Ans. 1 to 20.

(5) *With 20 lbs. and 40 sec. as units of mass and time respectively, find the unit of length that the derived unit of force may be equal to the weight of 1 lb. at a place where $g = 32.2$ ft.-per-sec. per sec.*

Ans. 2576 ft.

(6) *The unit of velocity being 20 cm. per sec., the unit of mass 15 grams, and the derived unit of force the weight of a kilogram, find the unit of time.*

Ans. $\frac{1}{3270}$ sec.

(7) *The value of a force expressed in dynes is to be expressed in absolute units of the meter-kilogram-minute system. By what number must it be multiplied?*

Ans. 0.036.

(8) *Show that the weight of one pound is equal to* 4.45×10^5 *dynes approximately.*

(9) *Show that* 1 *poundal is equivalent to* 13825 *dynes.*

(10) *With* 1 *ft. and* 1 *sec. as units of distance and time, find the unit of mass, in order that the derived unit of force may be equal to the weight of* 1 *lb. at a place where* $g = 32.16$ *ft.-per-sec. per sec.*

Ans. 32.16 lbs.

(11) *The unit of mass being* 20 *lbs., the unit of time* 1 *min., and the unit of length* 1 *yard, compare the derived unit of force with the poundal.*

Ans. 1 to 60.

(12) *Compare the values of the mass of a body as expressed in gravitation units of the ft.-lb.-sec. and yard-ton-min. systems* (ton = 2240 lbs.).

Ans. 2688000 to 1.

(13) *Show that the value of one dyne expressed in terms of the weight of one ton* (2240 *lbs.*) *is* 1003×10^{-12} *approximately.*

(14) *Reduce* 20 *poundals to absolute units of the yd.-cwt.-min. system* (1 cwt. = 112 lbs.).

Ans. $214\frac{2}{7}$ units.

(15) *Determine the unit of time in order that, the foot being the unit of length, the value of the intensity of gravity may be expressed by* 1 *instead of* g.

Ans. $\dfrac{1}{\sqrt{g}}$ sec.

(16) *The unit of acceleration being* 6 *ft.-per-sec. per sec., find* (*a*) *the unit of mass when the derived unit of force is equal to the weight of* 20 *lbs., and* (*b*) *the unit of force when the derived unit of mass is a mass of* 20 *lbs.* ($g = 32$).

Ans. (*a*) 107.2 lbs.; (*b*) 120 poundals or the weight of 3.73 lbs.

CHAPTER II.

DENSITY. SPECIFIC MASS. DETERMINATION OF SPECIFIC MASS.

Density.—The number of units of mass of a body divided by its number of units of volume, or the mass per unit of volume, is the **mean density** of the body.

The mean density gives then the number of pounds in a cubic foot, or the number of grams in a cubic centimeter.

The density *at a given point* of a body is the ratio of mass to volume of an indefinitely small portion of the body at that point. If this is the same at all points, the body is **homogeneous**, or the density is **uniform**. If it varies, the density is variable and the body is non-homogeneous.

The density of a body *in a given state* is the mass per unit of volume of any portion of the body in that state.

When the length of a body is great relatively to its other dimensions, the mass per unit of *length* is called its mean **linear density**.

For a thin body the mass per unit of *area* is called its mean **surface density**.

If m is the mass of a homogeneous body and V its volume and δ its density, we have

$$\delta = \frac{m}{V}$$

or density equals mass per unit of volume.

Unit of Density.—If $[M]$ is the unit of mass and m the number of units of mass, $[V]$ the unit of volume and V the number of units of volume, $[D]$ the unit of density and δ the number of units of density, we have

$$\delta[D] = \frac{m[M]}{V[V]}.$$

We shall have

$$\delta = \frac{m}{V}$$

provided we take

$$[D] = \frac{[M]}{[V]}.$$

The unit of density, then, is one unit of mass per unit of volume, as one pound per cubic foot, or one gram per cubic centimeter.

Specific Mass.—The density-ratio of a body relatively to that of some standard substance is properly called its **specific mass**. It is often called "specific gravity," as a consequence of not distinguishing between weight and mass. The ideas are different, but the

numerical values the same, since the weight of a body is proportional to its mass.

The standard substance taken is water. If γ is the density or mass of a unit of volume of water, and δ the density or mass of a unit of volume of any other body, then the specific mass ϵ is given by

$$\epsilon = \frac{\delta}{\gamma}. \qquad (1)$$

Since $\delta = \frac{m}{V}$, where m is the mass and V the volume of the body, we have

$$\epsilon = \frac{m}{\gamma V}. \qquad (2)$$

Since γ is the mass of a unit of volume of water, γV is the mass of a volume of water equal in volume to the body. Hence *the specific mass of any body is equal to the ratio of its mass to the mass of an equal volume of water.*

In the English system the mass of one cubic foot of pure water at 4° C., or the point of maximum density, is nearly 1000 ounces, or 62.5 lbs. (more exactly 998.6 ounces). The density of water is then about 62.5 lbs. per cubic foot, or

$$\gamma = \frac{62.5 \text{ lbs.}}{1 \text{ cub. ft.}}.$$

If then V is one cubic foot, we have, from (2),

$$\epsilon = \frac{m \text{ lbs.}}{62.5 \text{ lbs.}},$$

where m is the mass in pounds of one cubic foot of any body.

In the C. G. S. system, the mass of one cubic centimeter of pure water at 4° C. is very nearly one gram, and was intended to be so exactly. The density of water by this system is then

$$\gamma = \frac{1 \text{ gram}}{1 \text{ cub. c.}}.$$

If then V is one cubic centimeter, we have, from (2),

$$\epsilon = \frac{m \text{ grams}}{1 \text{ gram}},$$

where m is the mass in grams of one cubic centimeter. That is, the mass in grams of one cubic centimeter gives at once the specific mass, while in the English system the mass in pounds of one cubic foot must be divided by 62.5. Or inversely the specific mass of any body gives at once the mass in grams of one cubic centimeter of the body, while it must be multiplied by 62.5 to obtain the mass in pounds of one cubic foot.

Determination of Specific Mass.—A body totally immersed in water displaces its own volume of water. It is a well-known physical fact that a body so immersed is buoyed up by a force equal to the weight of the volume of water displaced.

If then a body is "weighed," i.e., its mass determined, and then weighed again while wholly immersed in water, the loss of weight in *gravitation units* gives the mass of the displaced water, or gives the mass of a volume of water equal to the volume of the body.

To determine the specific mass, then, we have only to *divide the weight of the body in gravitation units by its loss of weight in water in gravitation units.**

When very great accuracy is required the body should be weighed in a vacuum, or allowance must be made for the buoyant force of the air. But in all practical cases in mechanics this is an unnecessary refinement, and the weight in air may be taken as the measure of the true mass of the body.

Table of Specific Mass.—In the following table the density-ratios or specific masses, or so-called "specific gravity" with reference to water, of a few substances are given.

The exact value in any case will depend on the temperature and the mechanical process, such as hammering, etc., to which the bodies may have been subjected.

Air at 0° C	0.0012759	Tin	7.4
Alcohol at 0° C	0.791	Iron	7.7
Turpentine at 0° C	0.870	Copper	8.8
Ice	0.92	Silver	10.5
Sea-water at 0° C	1.026	Lead	11.4
Crown glass	2.5	Mercury at 0° C	13.596
Flint glass	3.0	Gold	19.3
Aluminum	2.6	Platinum	21.5
Zinc	7.0		

EXAMPLES.

(1) *The mass of a piece of limestone is* 310 *grams. When immersed in water it is balanced by a mass of* 188.5 *grams. What is the specific mass?*

Ans. Weight in air is $310g$ dynes. Weight in water is $188.5g$ dynes. Loss of weight is $310g - 188.5g = 121.5g$ dynes. Hence specific mass $= \dfrac{310g}{121.5g} = 2.55$.

(2) *In order to find the specific mass of a piece of oak, a piece of lead wire, which lost* 10.5 *grams when weighed in water, was wrapped around the wood, which weighed* 426.5 *grams. The compound mass was* 484.5 *grams lighter in the water than in the air. Find the specific mass.*

Ans. The loss of the wood alone was $484.5 - 10.5 = 474$. Hence specific mass $= \dfrac{426.5}{474} = 0.9$.

(3) *An iron vessel completely filled with mercury weighed* 500 *pounds, and lost when weighed in water* 40 *pounds. If the specific mass of the iron is* 7.2 *and of the mercury* 13.6, *find the mass of the vessel and of the mercury.*

Ans. Since specific mass $\epsilon = \dfrac{\delta}{\gamma}$, where δ is density and γ is density of water, and since $\delta = \dfrac{m}{v}$, where m is mass and v is volume, we have $\epsilon = \dfrac{m}{v\gamma}$, or $v = \dfrac{m}{\epsilon\gamma}$.

Let m_1 be the mass of the iron and m_2 the mass of the mercury, and m the combined mass.

* That is, we divide the number of units of mass of the body by the number of units of mass of an *equal volume of water*.

Then for the volume of the iron we have $v_1 = \dfrac{m_1}{\epsilon_1\gamma}$, for the volume of the mercury $v_2 = \dfrac{m_2}{\epsilon_2\gamma}$, and for the combined volume $v = \dfrac{m}{\epsilon\gamma}$. Hence we have

$$\frac{m_1}{\epsilon_1\gamma} + \frac{m_2}{\epsilon_2\gamma} = \frac{m}{\epsilon\gamma}, \text{ or } \frac{m_1}{\epsilon_1} + \frac{m_2}{\epsilon_2} = \frac{m}{\epsilon}.$$

Also $m_1 + m_2 = m$. Combining we have

$$m_1 = m \cdot \frac{\frac{1}{\epsilon} - \frac{1}{\epsilon_2}}{\frac{1}{\epsilon_1} - \frac{1}{\epsilon_2}}, \quad m_2 = m \cdot \frac{\frac{1}{\epsilon} - \frac{1}{\epsilon_1}}{\frac{1}{\epsilon_2} - \frac{1}{\epsilon_1}}.$$

In the present case we have $\epsilon = \dfrac{500}{40}$, $\epsilon_1 = 7.2$, $\epsilon_2 = 13.6$ and $m = 500$. Hence $m_1 = 49.54$ pounds, $m_2 = 450.46$ pounds.

NOTE.—This is called the *problem of Archimedes*, because first solved by him with reference to any alloy of gold and silver. Its application to alloys or chemical compositions is, however, limited, as in general in such cases there is a change of volume so that the combined volume is not equal to the sum of the volumes of the components.

(4) *In order to obtain the specific mass of rye in bulk, a bottle was filled with grains of rye well shaken together, and weighed. The weight of the bottle was found to be* 115 *grams when empty and* 235.75 *grams when filled with rye. When filled with water it weighed* 270.65 *grams. Find the specific mass of the grain.*

Ans. The weight of the grain is 120.75 grams, and the weight of an equal volume of water is 155.65 grams. Therefore specific mass $= \dfrac{120.75}{155.65} = 0.776$. A cubic foot of the grain weighs then $0.776 \times 62.5 = 48.5$ pounds.

(5) *To find the specific mass of a mixture, given the volume or mass, and specific mass, of each constituent.*

Ans. We must assume that the volume of a mixture is equal to the sum of the volumes of the constituents. This is not invariably the case, especially where there is chemical union.

Let m_1, m_2, m_3, etc., be the masses of the constituents;

ϵ_1, ϵ_2, ϵ_3, " " " specific masses of the constituents;

v_1, v_2, v_3, " " " volumes " " "

Let m, v and ϵ be the mass, volume and specific mass of the mixture. Let γ be the density or mass of a unit of volume of water.

Then $m_1 + m_2 + m_3 + \text{etc.} = m$. But $m_1 = \epsilon_1\gamma v_1$, $m_2 = \epsilon_2\gamma v_2$, etc. Hence

$$\epsilon_1 v_1\gamma + \epsilon_2 v_2\gamma + \epsilon_3 v_3\gamma + \text{etc.} = \epsilon v\gamma, \quad \text{or} \quad \epsilon = \frac{\epsilon_1 v_1 + \epsilon_2 v_2 + \epsilon_3 v_3 + \text{ etc.}}{v}$$

But $v = v_1 + v_2 + v_3 + \text{etc.}$ Therefore

$$\epsilon = \frac{\epsilon_1 v_1 + \epsilon_2 v_2 + \epsilon_3 v_3 + \text{etc.}}{v_1 + v_2 + v_3 + \text{etc.}} \quad \ldots\ldots\ldots \quad (1)$$

Again, we have $v_1 = \dfrac{m_1}{\epsilon_1\gamma}$, $v_2 = \dfrac{m_2}{\epsilon_2\gamma}$, etc. Hence

$$\frac{m}{\epsilon\gamma} = \frac{m_1}{\epsilon_1\gamma} + \frac{m_2}{\epsilon_2\gamma} + \frac{m_3}{\epsilon_3\gamma} + \text{etc.}$$

Therefore

$$\epsilon = \frac{m_1 + m_2 + m_3 + \text{etc.}}{\frac{m_1}{\epsilon_1} + \frac{m_2}{\epsilon_2} + \frac{m_3}{\epsilon_3} + \text{etc.}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(6) *Two equal vessels A and B are full and half full, respectively, of liquids of densities δ_1 and δ_2. If B is filled up from A and then A filled up from B, find the density of the mixture in A, the liquids being supposed to mix completely.*

Ans. $\dfrac{3\delta_1 + \delta_2}{4}$.

(7) *Three equal vessels A, B, C are half full of liquids of densities δ_1; δ_2, δ_3 respectively. If now B is filled up from A, and then C from B, find the density of the mixture in C, the liquids being supposed to mix completely.*

Ans. $\dfrac{\delta_1 + \delta_2 + 2\delta_3}{4}$.

(8) *To a salt solution whose specific mass is* 1.08 *and mass* 27 *ounces*, 4 *ounces of water are added. Find the specific mass of the mixture.*

Ans. $\frac{31}{29}$.

(9) *Find how much water must be added to* 27 *ounces of a salt solution whose specific mass is* 1.08, *in order that the specific mass of the mixture may be* 1.05.

Ans. 15 ounces.

(10) *When equal volumes of two substances are mixed, the specific mass of the mixture is* 3. *When equal weights are mixed the specific mass of the mixture is* $2\frac{2}{3}$. *Find the specific masses of the two substances.*

Ans. 2 and 4.

(11) *The masses and diameters of two spheres are as* 1 *to* 2. *Show that their densities are as* 4 *to* 1.

(12) *The diameter of the earth being* 1.275×10^9 *cm. and its density* 5.67 *times as great as that of water, find its mass.*

Ans. 6.15×10^{27} grams.

(13) *The linear density of a round bar of cast iron one inch in diameter is* 2.45 *lbs. per foot. Find the weight of a pipe* 2 *yards long, having a bore of* 16 *inches and a thickness of* $\frac{3}{4}$ *inch.*

Ans. 739 lbs.

(14) *A flat bar of iron* $4\frac{1}{2}$ *inches wide and* $\frac{5}{8}$ *inch thick has a linear density of* 9.91 *lbs. per ft. Find the weight of a bar of iron* 1 *inch square and* 1 *yard long.*

Ans. 10 lbs.

(15) *From the preceding example state a rule for finding the weight per foot of a bar of iron of any given constant area; also for finding the area if the weight per foot is given.*

Ans. To find the weight per foot in pounds, multiply the area in square inches by 10 and divide by 3.

To find the area in square inches, multiply the weight per foot by 3 and divide by 10.

(16) *The density of granite is* 160 *lbs. per cubic foot. A paving-block is* 4 *inches wide,* 9 *inches deep and* 12 *inches long. Find the number of tons* (2240 *lbs.*) *required to pave a street one mile long and* 20 *yards broad, allowing an interval of* 10 *per cent between the blocks.*

Ans. 15274 tons.

(17) *If the population of a country is* 35262762 *souls, and the area is* 120830 *square miles, what is the average "density" of the population?*

Ans. 292 inhabitants per square mile.

(18) *Find the specific mass of a piece of cork from the following data: Weight in air* 2 *grams, weight of cork and sinker in water* 4 *grams, weight of sinker in water* 12 *grams.*

Ans. 0.2.

(19) *A raft whose weight and specific mass are known floats in water. Show how to determine the greatest weight it can support without sinking.*

Ans. Let m be the mass and ϵ the specific mass of the raft. Then load $= \dfrac{m(1-\epsilon)}{\epsilon}$.

(20) *An empty balloon with its car and appendages weighs in air* 1200 *lbs. If a cubic foot of air weighs* $1\frac{1}{4}$ *oz., find how many cubic feet of gas must be used before the balloon will begin to ascend. Specific mass of the gas* 0.52, *compared to air.*

(21) *An iceberg has the form of a cube and floats flat with a height of* 30 *ft. above the ocean. Find the depth under water. Specific mass of ice* 0.92, *of sea-water* 1.026.

Ans. 260 feet.

(22) *Find the mass of the earth in tons* (2240 *lbs.*), *having given mean specific mass* 5.6, *mean radius* 4000 *miles.*

Ans. 6.16×10^{21} tons.

(23) *The unit of density being that of water, and the units of time and mass* 1 *minute and* 112 *lbs., find the magnitude of the derived unit of force.*

Ans. 0.0378 poundals.

(24) *The number of seconds in the unit of time being equal to the number of feet in the unit of length, the unit of force being the weight of* 750 *lbs.* ($g = 32$), *and a cubic foot of the standard substance having a mass of* 13500 *oz., find the unit of time.*

Ans. $5\frac{1}{3}$ sec.

CHAPTER III.

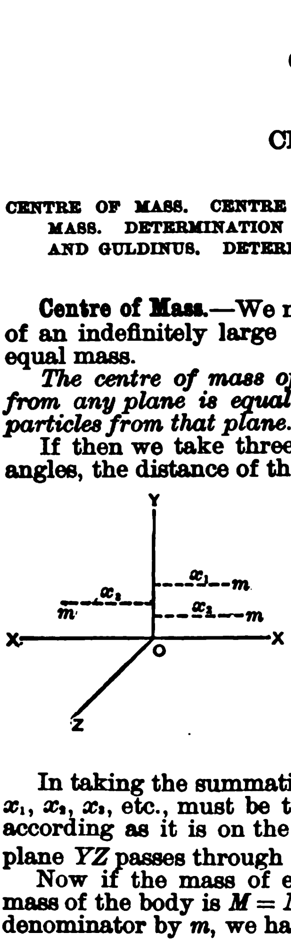
CENTRE OF MASS.

CENTRE OF MASS. CENTRE OF GRAVITY. PROPERTY OF THE CENTRE OF MASS. DETERMINATION OF CENTRE OF MASS. THEOREM OF PAPPUS AND GULDINUS. DETERMINATION OF CENTRE OF MASS BY CALCULUS.

Centre of Mass.—We may consider a material body as composed of an indefinitely large number of indefinitely small particles of equal mass.

The centre of mass of such a body is that point whose distance from any plane is equal to the average distance of all the equal particles from that plane.

If then we take three co-ordinate planes XY, YZ, ZX, at right angles, the distance of the centre of mass from each plane is equal to the average distance of all the equal particles from each plane.



Thus suppose a body composed of a number N of particles of equal mass. Let x_1, x_2, x_3, etc., be the distance of each particle from the co-ordinate plane YZ. Then we have for the average distance of all the particles, or for the distance $\bar{x}$ of the centre of mass from the plane YZ,

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \text{etc.}}{N} = \frac{\Sigma x}{N}.$$

In taking the summation $x_1 + x_2 + x_3 + \text{etc.} = \Sigma x$, each distance x_1, x_2, x_3, etc., must be taken with its appropriate sign $(+)$ or $(-)$ according as it is on the right or left of the plane YZ. If then the plane YZ passes through the centre of mass, $\bar{x} = 0$ and $\Sigma x = 0$.

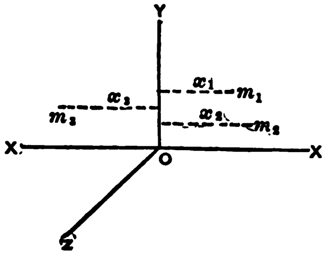
Now if the mass of each equal particle is m, the total mass or mass of the body is $M = Nm$. If then we multiply numerator and denominator by m, we have

$$\bar{x} = \frac{m\Sigma x}{M}.$$

If a material body is composed of particles of unequal mass, we may consider each of these particles as itself composed of particles of equal mass.

Thus suppose a body composed of particles whose masses are

m_1, m_2, m_3, etc. Let the first consist of a number n_1 of particles of equal mass m, the second of a number n_2 of particles of equal mass m, and so on. Then $m_1 = n_1 m$, $m_2 = n_2 m$, $m_3 = n_3 m$, etc. Let the entire number of equal particles be N, so that the total mass, or mass of the body, is $M = Nm$.



Then if x_1, x_2, x_3, etc., are the distances of the particles of unequal mass from the co-ordinate plane YZ, we have for the average distance of all the particles, or for the distance $\bar{x}$ of the centre of mass from the plane YZ,

$$\bar{x} = \frac{n_1 x_1 + n_2 x_2 + n_3 x_3 + \text{etc.}}{N}.$$

If we multiply numerator and denominator by m, we have

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \text{etc.}}{M} = \frac{\Sigma mx}{M}. \quad . \quad . \quad . \quad (1)$$

In the same way we have for the distance $\bar{y}$ of the centre of mass from the co-ordinate plane ZX

$$\bar{y} = \frac{\Sigma my}{M}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and for the distance $\bar{z}$ of the centre of mass from the co-ordinate plane XY

$$\bar{z} = \frac{\Sigma mz}{M}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We see then that the centre of mass of a body is such a point that if the number of units in the whole mass be multiplied by the number of units in the distance of this point from any plane, the result will be equal to the algebraic sum of the products obtained by multiplying the number of units in the mass of each elementary particle by the number of units in its distance from the same plane.

COR. In taking the sums of the products Σmx, Σmy, Σmz, for each elementary mass or particle, we must take x, y, z with their proper signs.

If then we take the origin of co-ordinates at the centre of mass, we have $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$; hence

$$\Sigma mx = 0, \quad \Sigma my = 0, \quad \Sigma mz = 0.$$

If we take polar co-ordinates and take the pole at the centre of mass, we have

$$\Sigma mr = 0,$$

where r is the distance of any particle from the pole.

That is, the algebraic sum of the moments of the masses (page 19) of all the particles with reference to the centre of mass is zero.

Centre of Gravity.—We shall see hereafter (page 75) that the centre of mass of a body *coincides with the point of application of the resultant of that system of parallel forces which acts upon all*

the particles of a translating body; that is, when each parallel particle force causes in the particle on which it acts the same acceleration in the same direction.

The earth's attraction for a body is the resultant of a system of forces acting upon the particles of the body, each particle force being directed towards the centre of the earth, and causing in the particle on which it acts an acceleration of the same magnitude. We have thus a system of forces not strictly parallel, but causing in each particle an acceleration of the same magnitude.

But practically the deviation from parallelism is insignificant, since the longest dimension of any body on the earth with which we have to deal is insignificant in comparison with the radius of the earth. Hence the accelerations are practically parallel as well as equal and *the resultant force of gravity upon a body passes practically through the centre of mass.* This resultant is the weight of the body. *The weight of a body acts practically, therefore, at the centre of mass.*

The centre of mass is therefore often called the "*centre of gravity.*" The term is, however, strictly speaking, incorrect. The term "centre of gravity" can only be properly applied to that point at which, if the entire mass of the body were concentrated, this point would attract and be attracted in all positions of the body, just the same as the body itself. In this sense, as we shall see (page 47), only a few bodies possess a centre of gravity, while all bodies have a centre of mass.

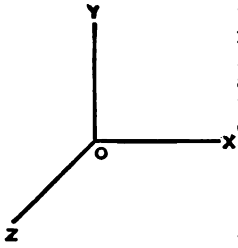
Centre of mass then has nothing to do with gravity. Gravity furnishes only a convenient practical method of locating it. The two ideas are entirely distinct.

Property of the Centre of Mass.—The importance of the centre of mass of a body, in Dynamics, depends on a property of it which we shall prove hereafter (page 83).

This property is as follows:

Whatever the motion of a rigid body may be, the centre of mass of the body moves precisely the same as if the body were replaced by a particle of equal mass at the centre of mass, and all the forces acting upon the body were transferred to this particle, without change in direction or magnitude. (For other properties of the centre of mass see page 75).

Determination of Centre of Mass.—We have just seen that the centre of mass of a body is such a point that if the number of units in the whole mass be multiplied by the number of units in the distance of this point from any plane, the result will be equal to the algebraic sum of the products obtained by multiplying the number of units in the mass of each elementary particle by the number of units in its distance from the same plane.



If we denote the volumes of the indefinitely small elements of a body by v_1, v_2, v_3, etc., and their densities by δ_1, δ_2, δ_3, etc., then the masses of these elements will be given by $m_1 = \delta_1 v_1$, $m_2 = \delta_2 v_2$, $m_3 = \delta_3 v_3$, etc. (page 10).

If then x_1, x_2, x_3, etc., are the distances, from the co-ordinate plane YZ, y_1, y_2, y_3, etc., from the co-ordinate plane ZX, z_1, z_2, z_3, etc., from the co-ordinate plane XY, we have for the co-ordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of the centre of mass in general

$$\left.\begin{aligned} \bar{x} &= \frac{\delta_1 v_1 x_1 + \delta_2 v_2 x_2 + \text{etc.}}{\delta_1 v_1 + \delta_2 v_2 + \text{etc.}}; \\ \bar{y} &= \frac{\delta_1 v_1 y_1 + \delta_2 v_2 y_2 + \text{etc.}}{\delta_1 v_1 + \delta_2 v_2 + \text{etc.}}; \\ \bar{z} &= \frac{\delta_1 v_1 z_1 + \delta_2 v_2 z_2 + \text{etc.}}{\delta_1 v_1 + \delta_2 v_2 + \text{etc.}}. \end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

If the body is *homogeneous*, we have $\delta_1 = \delta_2 = \delta_3$, etc. Hence if V is the volume of the body, we have for a *homogeneous body*,

$$\left.\begin{aligned} \bar{x} &= \frac{v_1 x_1 + v_2 x_2 + \text{etc.}}{v_1 + v_2 + \text{etc.}} = \frac{\Sigma vx}{V}; \\ \bar{y} &= \frac{v_1 y_1 + v_2 y_2 + \text{etc.}}{v_1 + v_2 + \text{etc.}} = \frac{\Sigma vy}{V}; \\ \bar{z} &= \frac{v_1 z_1 + v_2 z_2 + \text{etc.}}{v_1 + v_2 + \text{etc.}} = \frac{\Sigma vz}{V}. \end{aligned}\right\} \quad \ldots\ldots \quad (2)$$

Equations (1) and (2) give the position of the centre of mass for volumes, non-homogeneous or homogeneous.

For surfaces or areas we can put a for v and A for V, where a is the area of an element and A the entire area, and δ the *surface* density (page 10).

For lines we can put l for v and L for V, where l is the length of an element and L the entire length, and δ is the *linear* density (page 10).

Material Line, Area and Volume.—There is of course a certain inconsistency in speaking of the centre of mass of geometrical lines, areas and volumes, since they have no mass. The expression is, however, allowable, since we are understood to mean a *physical* or *material* line whose cross-section is constant and therefore cancels out of equations (1) and (2), δ being then the linear density; or a material area whose thickness is constant and therefore cancels out, δ being the surface density; or a volume filled with matter of uniform density, in which case δ cancels out and we have equations (2).

Moment of Mass, Volume, Area.—We may call the product of the magnitude of a mass, volume or area by the magnitude of the distance of its centre of mass from any plane or axis, the magnitude of the *moment* of the mass, volume or area, relatively to that plane or axis.

We can then express equations (1) and (2) by saying that the moment of the total mass, volume or area of a body with reference to any plane or axis is equal to the sum of the moments of the elementary masses, volumes or areas.

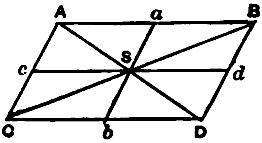
Plane and Axis of Symmetry.—A body is symmetrical with respect to a plane when the lines joining its particles, two and two, are parallel and bisected by the plane. In such case the centre of mass is in the plane and the equations for $\bar{x}$ and $\bar{y}$ are sufficient.

A body is symmetrical with respect to an axis when it is symmetrical with respect to two planes passing through that axis. In such case the centre of mass is in the axis and the equation for $\bar{x}$ is sufficient.

If a body is symmetrical with respect to two axes, the centre of mass is at their intersection. This point is then the *centre of figure.*

Many cases are simplified by the application of this principle of symmetry.

Thus the centre of mass of a homogeneous straight line is at the middle of the line; of a homogeneous circle or circular area or sphere, at the centre. For a parallelogram *ABCD* the line *ab* through the middle points of the sides *AB*, *CD*, bisects all lines parallel to those sides and is therefore an axis of symmetry. So is *cd* through the middle points of *AC*, *BD*. The diagonal *AD* bisects all lines parallel to the other diagonal and is an axis of symmetry. So is the diagonal *BC*. The surface would balance on a knife-edge along either of these lines. The centre of mass is then at *S*, their point of intersection.



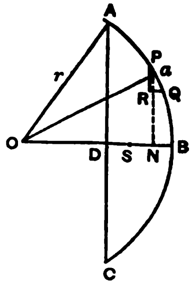
We shall make constant use of this principle of symmetry.

Centre of Mass of Homogeneous Material Lines.

(1) *Centre of Mass of Homogeneous Straight Line.*—The centre of mass of a homogeneous straight line is, by the principle of symmetry, at its middle point. For the line itself is one axis of symmetry, and a line at right angles to it at its middle point is another.

(2) *Homogeneous Circular Arc.*—The centre of mass for a homogeneous circular arc, if the arc is a full circle, is, by the principle of symmetry, at its centre of figure, or at the centre of the circle, because any diameter is an axis of symmetry. For any homogeneous arc in general, we may find the position of the centre of mass as follows:

Let *ABC* be a homogeneous circular arc with centre at *O*. Take the origin at *O* and let the axis of *X* pass through *O* and the centre *B* of the arc.



Then *OB* is an axis of symmetry, and the centre of mass *S* is on this axis. Let the chord $AC = c$, and the length of the arc *ABC* be *L*, and $r =$ radius. Take an indefinitely small element *PQ* whose length is *l* and whose centre of mass is at *a*, and let *PR* be the vertical projection of *PQ*.

Then we have by similar triangles

$$l : PR :: r : ON,$$

or, since $ON = x$, the moment of the mass of the element *PQ* with reference to an axis through *O* parallel to *AC* is proportional to $lx = rPR$. The sum of the projections *PR* of all the elements is $AC = c$. Hence the sum of the moments of all the elements is proportional to $\Sigma lx = r\Sigma PR = rc$. Since the entire length is *L*, we have from equation (2), page 19,

$$\bar{x} = \frac{\Sigma lx}{L} = \frac{rc}{L}.$$

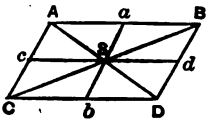
Therefore the centre of mass *S* of a circular arc *ABC* is on the axis of symmetry *OB* at a distance $\bar{x} = OS$ from the centre of the arc, which is a fourth proportional to the arc, the radius and the chord, or

$$L : r :: c : \bar{x}.$$

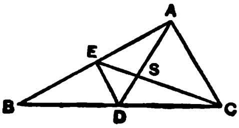
For a semicircle, $c = 2r$ and $L = \pi r$, hence $\bar{x} = \frac{2r}{\pi}$. For an entire circle, $c = 0$ and $\bar{x} = 0$, or the centre of mass is at the centre of figure.

Centre of Mass of Homogeneous Areas.

(3) *Homogeneous Parallelogram.*—Every line of the homogeneous parallelogram $ABCD$ parallel to AB or CD is bisected by the line ab drawn through the centres of the sides $ABCD$. Hence ab is an axis of symmetry. So is the line cd, or AD or BC. The centre of mass is then, by the principle of symmetry, at the centre of figure, or at the intersection S of the diagonals, or of the lines drawn between the middle points of opposite sides.



(4) *Homogeneous Triangle.*—Every line of the homogeneous triangle ABC parallel to BC is bisected by the line AD drawn from the vertex A to the centre D of the opposite side.



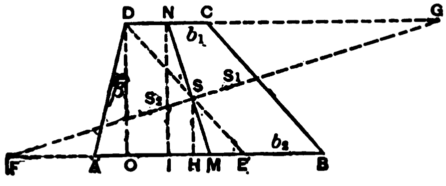
Hence AD is an axis of symmetry. So also is the line CE drawn from the vertex C to the centre E of the opposite side. The centre of mass is then at S. Since E and D are the middle points of AB, BC, and therefore DE is parallel to CA and equal to $\frac{1}{2}CA$, the triangles AFC and DFE are similar, and

$$DS : SA :: DE : AC \text{ or } :: 1 : 2.$$

Hence the centre of mass is on the line DA at a distance from D equal to $\frac{1}{3}DA$. In general the centre of mass is on the line from any vertex to the middle of the opposite side, at a distance from the vertex of $\frac{2}{3}$ the length of this line.

(5) *Homogeneous Trapezoid.*—We can determine the centre of mass of a homogeneous trapezoid as follows:

The line MN which joins the centres of the two bases AB and CD is an axis of symmetry, and the centre of mass S is on this line.



Denote the base AB by b_2 and CD by b_1, and the altitude DO by h. If we draw DE parallel to the side BC, we have a parallelogram $BCDE$ whose area is b_1h and the distance of whose centre of mass S_1 from AB is $\frac{h}{2}$, and a triangle ADE whose area is $\frac{(b_2 - b_1)h}{2}$ and the distance of whose centre of mass S_2 from AB is $\frac{h}{3}$.

The area of the trapezoid is $(b_1 + b_2)\frac{h}{2}$. If $\bar{y}$ is the distance HS of the centre of mass of the trapezoid from AB, we have

$$(b_1 + b_2)\frac{h}{2} \cdot \bar{y} = b_1h \cdot \frac{h}{2} + \frac{(b_2 - b_1)h}{2} \cdot \frac{h}{3} = (b_2 + 2b_1)\frac{h^2}{6}.$$

Hence

$$\bar{y} = HS = \frac{b_2 + 2b_1}{b_1 + b_2} \cdot \frac{h}{3}.$$

We have also

$$\frac{HM}{\bar{y}} = \frac{IM}{h}, \quad \text{or} \quad HM = \frac{\bar{y}}{h} IM.$$

Let the angle $ADO = \beta$, then

$$AO = h \tan \beta, \quad \text{and} \quad IM = \frac{b_2}{2} - h \tan \beta - \frac{b_1}{2}.$$

Therefore

$$HM = \frac{b_2 + 2b_1}{3(b_1 + b_2)} \left(\frac{b_2 - b_1}{2} - h \tan \beta \right).$$

If $\bar{x}$ is the distance AH of the centre of mass from A, we have, if $AO = a = h \tan \beta$,

$$\bar{x} = AH = \frac{b_2}{2} - \frac{b_2 + 2b_1}{3(b_1 + b_2)} \left(\frac{b_2 - b_1}{2} - h \tan \beta \right)$$
$$= \frac{b_1^2 + b_1 b_2 + b_2^2 + a(b_2 + 2b_1)}{3(b_1 + b_2)}.$$

We have also

$$\frac{MS}{\bar{y}} = \frac{NM}{h}, \quad \text{and} \quad \frac{NS}{h - \bar{y}} = \frac{NM}{h}.$$

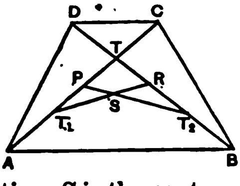
Hence

$$MS = \frac{b_2 + 2b_1}{b_1 + b_2} \cdot \frac{NM}{3}, \quad \text{and} \quad NS = \frac{2b_2 + b_1}{b_1 + b_2} \cdot \frac{NM}{3}.$$

Therefore

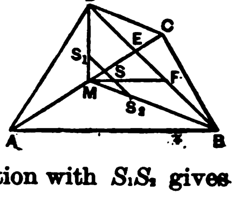
$$\frac{MS}{NS} = \frac{b_2 + 2b_1}{2b_2 + b_1} = \frac{\frac{1}{2}b_2 + b_1}{b_2 + \frac{1}{2}b_1} = \frac{AM + DC}{AB + NC} = \frac{AM + AF}{GC + NC}.$$

1st Construction.—If then we lay off $AF = DC$, and $CG = AB$, and join FG, the intersection S of FG with NM gives the centre of mass.



2d Construction.—Another convenient construction is as follows: Draw the diagonals AC, BD, intersecting at T. Lay off along AC the distance $AT_1 = CT$ and along BD the distance $BT_2 = DT$. Bisect the diagonals at R and P and join RT_1 and PT_2. The intersection S is the centre of mass. Student will prove.

(6) *Homogeneous Trapezium.*—In order to find the centre of mass of any homogeneous four-sided area $ABCD$, we can divide it by means of a diagonal AC into two triangles and determine their centres of mass S_1 and S_2 by (4). We thus obtain a line S_1S_2. If we again divide the area by the diagonal BD into two other triangles and determine their centres of mass, we obtain a second line whose intersection with S_1S_2 gives the centre of mass S of the whole area.



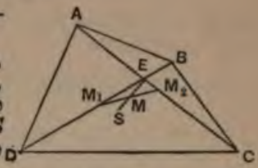
We can, however, proceed more simply by bisecting the diagonal AC at M and laying off the longer segment BE of the other diagonal from D to F so that $DF = BE$.

Then draw FM and take $MS = \frac{1}{3}FM$. Then S is the centre of mass.

For we have $MS_1 = \frac{1}{3}MD$ and $MS_2 = \frac{1}{3}MB$, hence S_1S_2 is parallel to BD. But $SS_1 \times$ area $ACD = SS_2 \times$ area ACB, or $SS_1 \times DE = SS_2 \times BE$, whence $SS_1 : SS_2 :: BE : DE$. But we have by construction $BE = DF$, and $DE = BF$; hence $SS_1 : SS_2 :: DF : BF$. Hence MF cuts S_1S_2 at the centre of mass S.

1*st Construction.*—We have then the following construction:

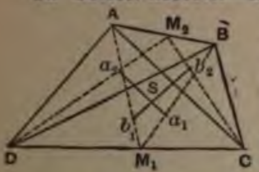
Bisect one diagonal AC at M. Lay off the longer segment BE of the other diagonal from D to F, so that $DF = BE$. Then join MF and take $MS = \frac{1}{3}MF$. Then S is the centre of mass.



2*d Construction.*—We have also the following construction:

Let E be the intersection of the diagonals, and M_1, M_2 their middle points. Join M_1, M_2, and let M be its middle point. Draw the line EM and produce it to S, so that MS equals one third of EM. Then S is the centre of mass. Student will prove.

3*d Construction.*—Draw the diagonal DB, dividing the figure into the two triangles DAB and BDC. The centres of mass a_2 and a_1 of each of these triangles are in the lines DM_2 and BM_1 drawn from the vertices D and B to the middle points M_2 and M_1 of the opposite sides, so that $Da_2 = \frac{2}{3}DM_2$ and $Ba_1 = \frac{2}{3}BM_1$. The centre of mass is then in the line a_2a_1.



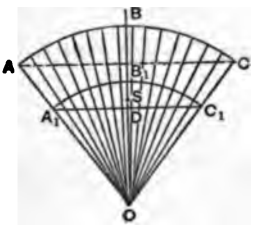
Now draw the diagonal CA, dividing the figure into the two triangles CAB and ADC. The centres of mass b_2 and b_1 of each of these triangles are in the lines CM_2 and AM_1, so that $Cb_2 = \frac{2}{3}CM_2$ and $Ab_1 = \frac{2}{3}AM$. The centre of mass is then in the line b_1b_2. The centre of mass S is then at the intersection of a_2a_1 and b_2b_1.

(7) *Homogeneous Plane Polygon.*—To find the centre of mass of any homogeneous plane polygon, we can divide it into triangles, consider the area of each triangle concentrated at its centre of mass, and find the moments of each with reference to two rectangular axes.

A convenient and sufficiently accurate method which is often employed is to draw the polygon to scale upon stiff manilla paper. Then cut the area out and balance it in two positions upon a knife-edge. Two axes of symmetry are thus determined, and the centre of mass of the area is at their intersection.

A similar method may be employed for finding the *area* of an irregular figure. Draw the area upon paper. Measure carefully the area of the sheet and *weigh* it in a delicate laboratory balance. Then cut the area out and weigh it. The areas are as their weights.

(8) *Homogeneous Circular Sector.*—The centre of mass of a homogeneous circular sector ACO coincides with the centre of mass S of the arc $A_1B_1C_1$ which has the same central angle and whose radius OA_1 is two thirds that of the sector OA. For the sector can be divided into an indefinite number of small triangles, the centre of mass of each of which is at a distance from O of two thirds of the radius. These centres give the arc $A_1B_1C_1$.



The centre of mass S of the sector lies, therefore, upon the radius of symmetry OB which bisects this arc $A_1B_1C_1$, and at a distance OS from the centre (page 20) given by

$$OS = \frac{\text{chord } A_1C_1}{\text{arc } A_1B_1C_1} \cdot \frac{2}{3} OA = \frac{4}{3} \cdot \frac{\sin \frac{\theta}{2}}{\theta} \cdot r,$$

where r denotes the radius of the sector and θ the central angle AOC in radians.

For the semicircle $\theta = \pi$, $\sin \frac{\theta}{2} = 1$ and

$$OS = \frac{4}{3\pi} r = \frac{14}{33} r, \text{ approximately.}$$

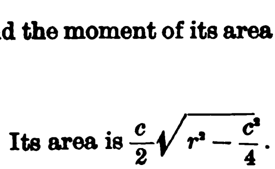
For a quadrant

$$OS = \frac{4\sqrt{2}}{3\pi} r = 0.6002r.$$

For a sextant

$$OS = \frac{2}{\pi} r = 0.6366r.$$

(9) *Homogeneous Segment of a Circle.*—The centre of mass of the homogeneous segment of a circle ABC is in the radius of symmetry OB and may be found by placing the moment of its area relative to an axis through O parallel to AC equal to the difference of the moments of the areas of the sector $ABCO$ and of the triangle ACO.



Let r be the radius OA, c the chord AC and A the area of the segment ABC, and l the length of arc ABC. Then the area of the sector is $\frac{rl}{2}$. The distance OS_1 for the centre of mass of the sector is $\frac{c}{l} \cdot \frac{2}{3} r$, and the moment of its area is $\frac{cr^2}{3}$.

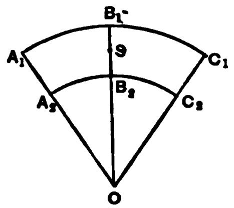
The height of the triangle is $\sqrt{r^2 - \frac{c^2}{4}}$. Its area is $\frac{c}{2}\sqrt{r^2 - \frac{c^2}{4}}$. The distance OS_2 for the centre of mass is $\frac{2}{3}\sqrt{r^2 - \frac{c^2}{4}}$. The moment of the area of the triangle is then $\frac{cr^2}{3} - \frac{c^3}{12}$. Hence we have

$$A \,.\, OS = \frac{1}{3} cr^2 - \left(\frac{cr^2}{3} - \frac{c^3}{12}\right) = \frac{c^3}{12}, \quad \text{or} \quad OS = \frac{c^3}{12A}.$$

That is, the centre of mass of a segment of a circle is on the radius of symmetry OB, at a distance OS from the centre of the circle equal to the cube of the chord AC divided by 12 times the area of the segment.

For a semicircular segment, $c = 2r$ and $A = \frac{\pi r^2}{2}$ and $CS = \frac{4r}{3\pi}$, as we have already found it (8).

(10) *Homogeneous Circular Ring.*—The centre of mass of a homogeneous circular ring can now be found. It is in the radius of symmetry OB_1. The area of the ring is the difference of area of two sectors $OA_1B_1C_1$ and $OA_2B_2C_2$. If $OA_1 = r_1$ and $OA_2 = r_2$ and the chords $A_1C_1 = c_1$, $A_2C_2 = c_2$, we have the moments of the areas of the sectors relative to an axis through O parallel to A_1C_1 equal to $\frac{c_1r_1^2}{3}$ and $\frac{c_2r_2^2}{3}$. The area of the ring is



$$\frac{r_1^2\theta}{2} - \frac{r_2^2\theta}{2} = \theta\left(\frac{r_1^2 - r_2^2}{2}\right),$$

where θ is the central angle A_1OC_1 in radians. Hence, since $\frac{c_2}{c_1} = \frac{r_2}{r_1}$,

$$\theta\left(\frac{r_1^2 - r_2^2}{2}\right) \cdot OS = \frac{c_1}{3r_1}\left(r_1^3 - r_2^3\right), \quad \text{or} \quad OS = \frac{r_1^3 - r_2^3}{r_1^2 - r_2^2} \cdot \frac{2c_1}{3r_1\theta}.$$

If l is the length of the arc $A_1B_1C_1$, this becomes

$$OS = \frac{2c_1}{3l}\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right),$$

or, since $l = r_1\theta$ and $c_1 = 2r_1 \sin\frac{\theta}{2}$,

$$OS = \frac{4\sin\frac{\theta}{2}}{3\theta} \cdot \frac{r_1^3 - r_2^3}{r_1^2 - r_2^2} = \frac{\sin\frac{1}{2}\theta}{\theta}\left[1 + \frac{1}{12}\left(\frac{b}{R}\right)^2\right]2R,$$

where $b = r_1 - r_2$ and $R = \frac{r_1 + r_2}{2}$.

(11) *Surface of a Cylinder.*—The centre of mass of the *homogeneous* surface of a cylinder lies at the centre of its axis. For all the equal-circle elements of the surface obtained by taking slices parallel to the base have their centres and centres of mass upon this axis. At these centres of mass the mass of each element may be concentrated. The centre of mass of the cylindrical surface is then the centre of mass of the axis.

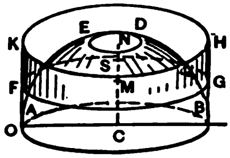
For the same reason the centre of mass of the surface of a prism lies in the middle of the line which unites the centres of mass of its bases.

(12) *Surface of a Right Cone.*—The centre of mass of the homogeneous surface of a right cone lies in the axis of the cone at two thirds of its length from the apex. For the curved surface can be divided into an indefinite number of small triangles. The centres of mass of all these triangles form a circle which is situated at a distance of two thirds of the axis from the apex, and whose centre of mass lies in the axis.

The same holds true for a right pyramid.

(13) *Surface of a Spherical Segment, Zone or Hemisphere.*—The centre of mass of the homogeneous surface of a spherical segment or zone or hemisphere is at the middle of its axis or height.

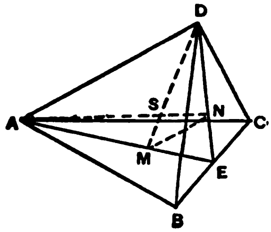
For, according to Geometry, the spherical zone $ABDE$ has the same area as the surface $FGHK$ of a cylinder whose height is equal to the height MN of the zone and whose radius is the radius CO of the sphere. This holds for all ring-shaped elements obtained by passing planes parallel to the base through the zone. Hence the centre of mass for the surface of the spherical zone, segment or hemisphere is at the middle S of its height MN and coincides with that of the cylinder.



Centre of Mass of Volumes.

(14) *Volume of a Homogeneous Prism.*—The centre of mass for a solid homogeneous prism is at the middle of its axis, or the line joining the centres of mass of its two bases. For by passing planes parallel to the bases we divide it into equal slices whose centres of mass lie in the axis.

(15) *Homogeneous Pyramid and Cone.*—Let $ABCD$ be a homogeneous triangular pyramid. Take E at the middle point of BC and draw AE and DE. Let $ME = \frac{1}{3}AE$, and $EN = \frac{1}{3}DE$.



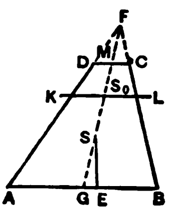
Draw DM and AN. Then DM and AN are axes of symmetry, and the centre of mass is at their intersection S. But MN must be parallel to AD and equal to $\frac{1}{3}AD$, and the triangle MNS is similar to DAS. Hence $MS = \frac{1}{3}DS$, or $DS = 3MS$; and $MD = 4\,MS$, or $MS = \frac{1}{4}MD$.

The centre of mass for the pyramid is then on the line joining a vertex with the centre of mass of the opposite base. at a distance from the vertex of three fourths the length of this line.

Since every pyramid and cone is composed of triangular pyramids with a common vertex, the centre of mass of any pyramid or cone is in the line joining the apex with the centre of mass of the base, at a distance from the vertex of three fourths the length of this line, or at a vertical distance of three fourths the altitude.

We can therefore determine the centre of mass of a pyramid or cone by passing a plane through the body parallel to the base at a distance of three fourths the altitude from the vertex, and finding the centre of mass of this section.

(16) *Frustum of a Cone or Pyramid.*—The centre of mass of a homogeneous frustum of a cone or pyramid lies in the line GM joining the centres of mass of the two parallel bases. If we denote by A_1 the area of the base AB, and by A_2 the area of the base DC, and by h the altitude between them, the height x of the point F above DC is given by



$$\frac{A_1}{A_2} = \frac{(h+x)^2}{x^2}, \quad \text{or} \quad x = \frac{h\sqrt{A_2}}{\sqrt{A_1} - \sqrt{A_2}},$$

and

$$x + h = \frac{h\sqrt{A_1}}{\sqrt{A_1} - \sqrt{A_2}}.$$

The moment of the entire pyramid with reference to its face is

$$\frac{A_1(x+h)}{3}\cdot\frac{x+h}{4}=\frac{1}{12}\cdot\frac{h^2A_1^2}{(\sqrt{A_1}-\sqrt{A_2})^2},$$

and that of the part of the pyramid which is wanting is

$$\frac{A_2x}{3}\left(h+\frac{x}{4}\right)=\frac{1}{3}\cdot\frac{h^2\sqrt{A_2}^3}{\sqrt{A_1}-\sqrt{A_2}}+\frac{1}{12}\cdot\frac{h^2A_2^2}{(\sqrt{A_1}-\sqrt{A_2})^2}.$$

Hence the moment of the truncated pyramid is found by subtracting the second from the first, after reduction, to be

$$\frac{h^2}{12}\left(A_1+2\sqrt{A_1A_2}+3A_2\right).$$

The volume of the frustum is $\left(A_1+\sqrt{A_1A_2}+A_2\right)\frac{h}{3}$. Therefore the distance of the centre of mass S from the base is

$$SE=\frac{A_1+2\sqrt{A_1A_2}+3A_2}{A_1+\sqrt{A_1A_2}+A_2}\cdot\frac{h}{4}.$$

The distance S_0S of this point from the plane KL passing through the middle of the body parallel to the base and dividing the altitude into two equal parts is

$$S_0S=\frac{h}{2}-SE=\frac{A_1-A_2}{(A_1+\sqrt{A_1A_2}+A_2)}\cdot\frac{h}{4}.$$

If the radii of the bases of a frustum of a cone are r_1 and r_2, we have

$$A_1=\pi r_1^2,\quad A_2=\pi r_2^2,$$

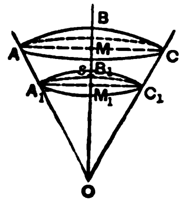
and

$$SE=\frac{r_1^2+2r_1r_2+3r_2^2}{r_1^2+r_1r_2+r_2^2}\cdot\frac{h}{4};$$

$$S_0S=\frac{r_1^2-r_2^2}{r_1^2+r_1r_2+r_2^2}\cdot\frac{h}{4}.$$

(17) *Spherical Sector.*—If the homogeneous circular sector AOB is revolved about its radius OB, a homogeneous spherical sector AOC is generated.

We can consider this body as composed of an indefinite number of pyramids, whose common apex is at O and whose bases form the spherical zone ABC. The centres of mass of each of these pyramids are at a distance of three fourths of the radius OB of the sphere from O, and they form a second spherical zone $A_1B_1C_1$, whose radius $OB_1=\frac{3}{4}OB$.



The centre of mass of this zone is then the centre of mass of the spherical sector. If we put $OA=OC=r$, and the altitude BM of the exterior zone $=h$, we have

$$OB_1=\frac{3}{4}r\quad\text{and}\quad B_1M_1=\frac{3}{4}h.$$

Hence, by (13),

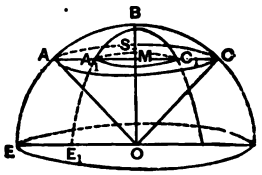
$$SB_1 = \frac{1}{2}M_1B_1 = \frac{3}{8}h,$$

and the distance of the centre of mass of the spherical sector from the centre O is

$$OS = OB_1 - SB_1 = \frac{3}{4}r - \frac{3}{8}h = \frac{3}{4}\left(r - \frac{h}{2}\right).$$

For a hemisphere, $r = h$, and $OS = \frac{3}{8}r$, or the centre of mass of a hemisphere is on its radius of symmetry at a distance of $\frac{3}{8}$ this radius from the centre.

(18) *Spherical Segment or Spheroid.*—We may obtain the centre of mass for a homogeneous spherical segment by putting the moment of the segment equal to that of the spherical sector $ABCO$ less that of the cone ACO.



Denoting again the radius OB of the sphere by r, and the altitude BM by h, we have the moment of the sector

$$= \frac{2}{3}\pi r^2 h \, . \, \frac{3}{4}\left(r - \frac{h}{2}\right) = \frac{1}{4}\pi r^2 h(2r - h),$$

and that of the cone

$$= \frac{1}{3}\pi h(2r - h)(r - h) \, . \, \frac{3}{4}(r - h) = \frac{1}{4}\pi h(2r - h)(r - h)^2.$$

Hence the moment of the segment is

$$V \times OS = \frac{1}{4}\pi h(2r - h)[r^2 - (r - h)^2] = \frac{1}{4}\pi h^2(2r - h)^2.$$

The volume of the segment is $V = \frac{1}{3}\pi h^2(3r - h)$, hence

$$OS = \frac{\frac{1}{4}\pi h^2(2r - h)^2}{\frac{1}{3}\pi h^2(3r - h)} = \frac{3}{4}\,\frac{(2r - h)^2}{3r - h}.$$

If we put $h = r$, the segment becomes a hemisphere, and, as before, $OS = \frac{3}{8}r$.

The result holds good for the segment A_1BC_1 of a spheroid generated by the resolution of the arc BA_1 of an ellipse about its major axis $OB = r$. For if we make $BM = x$ and $MA_1 = y$, the equation of the ellipse is

$$y^2 = \frac{b^2}{r^2}(2rx - x^2),$$

where $b = OE_1$. The equation of the circle is $y^2 = 2rx - x^2$. Hence $\frac{MA_1^2}{MA^2} = \frac{b^2}{r^2}$. We must then multiply not only the volume but also

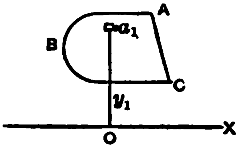
the moment of the spherical segment by $\frac{b^2}{r^2}$ to obtain the volume and moment of the segment of the spheroid. Therefore the quotient $OS = \frac{\text{moment}}{\text{volume}}$ is not changed.

In general, then, we have

$$OS = \frac{3}{4}\,\frac{(2r-h)^2}{3r-h},$$

where r denotes that semi-axis about which the ellipse is revolved when generating the spheroid.

Theorem of Pappus and Guldinus.—If a plane surface ABC is revolved about an axis OX, every element of it, as a_1, a_2, etc., describes a volume. If the distances of these elements from OX are y_1, y_2, etc., and the angle of rotation is θ radians, we have for the entire volume V generated



$$V = a_1y_1\theta + a_2y_2\theta + \ldots = \theta\Sigma ay.$$

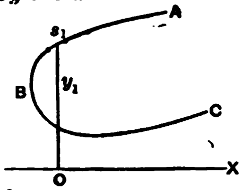
If $\bar{y}$ is the distance of the centre of mass of the surface ABC from OX, and A is its area, we have

$$A\bar{y} = a_1y_1 + a_2y_2 + \ldots = \Sigma ay.$$

Hence

$$A\bar{y}\theta = \theta\Sigma ay = V.$$

That is, *the volume generated by the revolution of a plane area which lies wholly on one side of the axis equals the area multiplied by the distance described by its centre of mass.*



In the same way, if a plane curve ABC is revolved about an axis OX, every element of it, as s_1, s_2, etc., describes a surface. The entire surface generated is

$$A = s_1y_1\theta + s_2y_2\theta + \ldots = \theta\Sigma sy.$$

If $\bar{y}$ is the distance of the centre of mass of the curve from OX, and L is the length of the curve, we have

$$L\bar{y} = s_1y_1 + s_2y_2 + \ldots = \Sigma sy.$$

Hence

$$L\bar{y}\theta = \theta\Sigma sy = A.$$

That is, *the area generated by the revolution of a line about a fixed axis equals the length of the line multiplied by the distance described by its centre of mass.*

These properties are known as the *theorems of Pappus and Guldinus.* By means of them, the volume, or the centre of mass, in many cases, may be very simply determined.

EXAMPLES.

(1) *The surface of a sphere is $4\pi r^2$, and the length of a semi-circumference is πr. Find the centre of mass for a semi-circle.*

Ans. On the radius of symmetry at a distance from the centre of $\frac{2r}{\pi}$. [See (2), page 20.]

(2) *The volume of a sphere is* $\frac{4}{3}\pi r^3$, *and the area of a semi-circle is* $\frac{1}{2}\pi r^2$. *Find the centre of mass of the surface of a semi-circle.*

Ans. On the radius of symmetry at a distance from the centre of $\frac{4r}{3\pi}$. [See (8), page 24.]

(3) *An ellipse revolves about a line in its plane, the perpendicular distance of which from the centre is equal to c. Find the volume of the ring generated by a complete revolution.*

Ans. Let a and b be the semi-axes of the generating ellipse. Then the generating area is $A = \pi ab$. The path described by the centre of mass is $2\pi c$. Hence the volume is $2\pi^2 abc$. This volume is the same whatever the position or direction of the axis of revolution with respect to the axes of the ellipse, provided that the perpendicular distance c from the centre to the axis is the same.

[**Determination of Centre of Mass by Calculus.**—When a body is of such form that we know the relations between its co-ordinates for any point, and its density is a function of the co-ordinates, we may write (1) and (2), page 19, in Calculus notation:

$$\bar{x} = \frac{\int \delta x dV}{\int \delta dV}, \quad \bar{y} = \frac{\int \delta y dV}{\int \delta dV}, \quad \bar{z} = \frac{\int \delta z dV}{\int \delta dV}, \quad . \quad . \quad (3)$$

where δ is the density for any elementary volume dV. If the body is homogeneous, δ is constant and $\int dV = V =$ the entire volume, and

$$\bar{x} = \frac{\int x dV}{V}, \quad \bar{y} = \frac{\int y dV}{V}, \quad \bar{z} = \frac{\int z dV}{V}. \quad . \quad . \quad . \quad (4)$$

From these equations the co-ordinates of the centre of mass are found by integrating between the limits which determine the volume.

From these general formulas we can readily deduce special formulas for special cases.

[**Centre of Mass of Lines.**—Thus if s is the length of a line and a its transverse section at any point, then ds is an element of length, and $dV = ads$, and (3) becomes

$$\bar{x} = \frac{\int a\delta x ds}{\int a\delta ds}, \quad \bar{y} = \frac{\int a\delta y ds}{\int a\delta ds}, \quad \bar{z} = \frac{\int a\delta z ds}{\int a\delta ds}. \quad . \quad . \quad (5)$$

If the line is homogeneous and the transverse section constant we have

$$\bar{x} = \frac{\int x ds}{s}, \quad \bar{y} = \frac{\int y ds}{s}, \quad \bar{z} = \frac{\int z ds}{s}. \quad . \quad . \quad (6)$$

If the line is a plane curve, we can take its plane that of xy. Then $\bar{z} = 0$, and the first two of (5) and (6) are insufficient. If the line is a straight line, we may take it coinciding with the axis of x. Then $\bar{y}$ and $\bar{z}$ are zero, $ds = dx$, and the first of (5) and (6) are sufficient.

EXAMPLES.

(1) *Find the center of mass of a homogeneous straight line.*

Ans. In this case we have $\bar{x} = \dfrac{\int_0^s x\,dx}{s} = \dfrac{s}{2}$, which is also evident from the principle of symmetry.

(2) *Find the center of mass of a straight fine wire of uniform section, in which the density varies directly as the distance from one end.*

Ans. If δ_1 is the density at a distance unity, and the axis of x coincides with the line, and the origin is taken at the end of the line, the density δ at any distance x is proportional to $\delta_1 x$, and $ds = dx$; hence from equation (5)

$$x = \frac{\int_0^s \delta_1 x^2\,dx}{\int_0^s \delta_1 x\,dx} = \frac{2}{3}s.$$

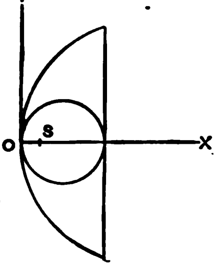
COR. If the density is constant but the section varies directly as the distance, we have the same result. The wire in this case would become a homogeneous triangular plate of uniform thickness. Hence the centre of mass of a *triangle* is on the axis of symmetry at a distance from the vertex of two thirds the length of that axis. [See (4), page 2.]

(3) *Find the center of mass of a straight fine wire of uniform section, in which the density varies as the square of the distance from one end.*

In this case we have

$$\bar{x} = \frac{\int_0^s \delta_1 x^3\,dx}{\int_0^s \delta_1 x^2\,dx} = \frac{3}{4}s.$$

COR. If the density is constant but the section varies as the square of the distance, we have the same result. The wire then becomes a homogeneous *cone* or *pyramid*, whether right or oblique, or whether the base be regular or irregular. [See (15), page 26.]



(4) *Find the center of mass of a homogeneous cycloid.*

Take the origin at O and let the axis OX be the axis of symmetry. Then if s is the length of the curve and r the radius of the generating circle, we have for the equation of the cycloid

$$s^2 = 8rx. \qquad \ldots \ldots \qquad (1)$$

Hence

$$ds = (2r)^{\frac{1}{2}}x^{-\frac{1}{2}}dx.$$

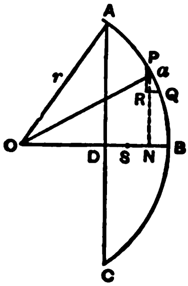
From equation (6), therefore,

$$\bar{x} = \frac{\int_0^x (2r)^{\frac{1}{2}}x}{(8rx)^{\frac{1}{2}}} = \frac{x}{3}.$$

When $x = 2r$, we have the curve corresponding to one complete revolution of the generating circle, and $\bar{x} = \frac{2}{3}r$. That is, the centre of mass for the curve is on the axis of symmetry at a distance OS from the vertex equal to one third of the diameter of the generating circle.

(5) *Find the centre of mass of a homogeneous circular arc.*

Let ABC be a circular arc, with centre at O. Take the origin at O and let the axis of x coincide with the axis of symmetry OB. Let $AC =$ chord $= c$, and the length of arc $ABC = s$, and $r =$ radius. Take an indefinitely small element $PQ = ds$, whose centre of mass is at a, so that $aN = y$ and the horizontal projection $QR = dx$.



Then

$$ds : dx :: r : y, \quad \text{or} \quad ds = \frac{rdx}{y}.$$

Hence $xds = \frac{rxdx}{y}$. But the equation of the circle is

$$x^2 + y^2 = r^2, \therefore xdx = -ydy,$$

and, therefore, $xds = -rdy$. From equation (6)

$$\bar{x} = \frac{\int_{+\frac{c}{2}}^{-\frac{c}{2}} -rdy}{l} = \frac{rc}{l}.$$

Hence the distance OS of the centre of mass from the centre of the circle is a fourth proportional to the arc, the radius, and the chord, or [see (2), page 20]

$$s : r :: c : \bar{x}.$$

[**Centre of Mass of Plane Surfaces.**—Let the plane of xy coincide with the surface. Then $\bar{z} = 0$. If we consider the surface as a thin material plate of density δ at any point and thickness τ, we have the elementary area $dxdy$ and the elementary volume $\tau dxdy = dV$, and equation (3), page 30, becomes

$$\bar{x} = \frac{\int\int \tau\delta xdxdy}{\int\int \tau\delta dxdy}, \quad \bar{y} = \frac{\int\int \tau\delta ydxdy}{\int\int \tau\delta dxdy}. \quad . \quad . \quad . \quad . \quad (7)$$

If τ is constant and the material homogeneous, or δ constant, we have the entire area

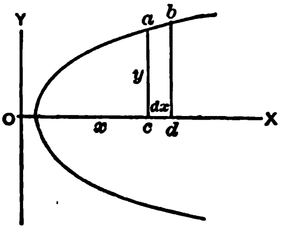
$$A = \int\int dxdy = \int xdy = \int ydx, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and

$$\bar{x} = \frac{\int yxdx}{A}, \quad \bar{y} = \frac{\frac{1}{2}\int y^2dx}{A}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If the axis of x is an axis of symmetry, $\bar{y} = 0$, and the value of $\bar{x}$ is sufficient.

The student will note that ydx is any elementary area $abdc$. Hence $\int ydx$ is the entire area A. Also $ydx \times x$ is the moment of the elementary area with reference to the axis of Y; and since the centre of mass of this area is at a distance $\frac{y}{2}$ above the axis of x, $ydx \times \frac{1}{2}y$ is its moment with reference to the axis of x. Hence we have equations (9).



If we take polar co-ordinates, we can replace dV in equations (8), page 30, by $\tau\rho d\rho d\theta$; and since $x = \rho\cos\theta$, $y = \rho\sin\theta$, where θ is the angle of the radius vector ρ with the horizontal, we obtain

$$\bar{x} = \frac{\int\int \tau\delta\rho^2 d\rho\cos\theta d\theta}{\int\int \tau\delta\rho d\rho d\theta}, \quad \bar{y} = \frac{\int\int \tau\delta\rho^2 d\rho\sin\theta d\theta}{\int\int \tau\delta\rho d\rho d\theta}. \quad . \quad . \quad (10)$$

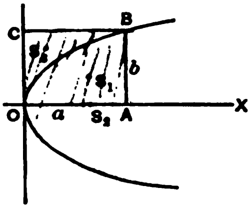
If the thickness is constant and the material homogeneous, τ and δ disappear and

$$\bar{x} = \frac{\int\int \rho^2 d\rho\cos\theta d\theta}{A}, \quad \bar{y} = \frac{\int\int \rho^2 d\rho\sin\theta d\theta}{A}. \quad . \quad . \quad . \quad (11)$$

EXAMPLES.

(1) *Find the centre of mass of a homogeneous semi-parabolic area whose length is a and height b.*

The equation of the parabola referred to the vertex is $y^2 = 2px$. When $x = a$, we have $y = b$; hence $2p = \frac{b^2}{a}$, and the equation becomes



$$y^2 = \frac{b^2}{a}x.$$

From equation (8)

$$A = \int_0^a ydx = \int_0^a \frac{b}{\sqrt{a}}x^{\frac{1}{2}}dx = \frac{2}{3}ab.$$

Therefore, from equation (9), we have for the distance of the centre of mass S_1 from O, upon OX,

$$\bar{x} = \frac{\int_0^a \frac{b}{\sqrt{a}}x^{\frac{3}{2}}dx}{\frac{2}{3}ab} = \frac{3}{5}a,$$

and for the distance above OX

$$\bar{y} = \frac{\frac{1}{2}\int_0^a \frac{b^2}{a} x\,dx}{\frac{2}{3}ab} = \frac{3}{8}b.$$

For the entire parabola we have two equal elementary areas $y\,dx$, one above and one below OX. The centre of mass S_2 is then in the axis of symmetry OX at a distance from the vertex

$$\bar{x} = \frac{2\int_0^a \frac{b}{\sqrt{a}} x^{\frac{3}{2}}\,dx}{2 \times \frac{2}{3}ab} = \frac{3}{5}a.$$

For the parabolic area OBC we have for origin at O the equation

$$y = b - b\sqrt{\frac{x}{a}}, \quad \text{and} \quad A = \frac{1}{3}ab.$$

Hence, from equation (9), we have for the centre of mass S_3

$$\bar{x} = \frac{\int_0^a bx\,dx - \frac{b}{\sqrt{a}} x^{\frac{3}{2}}\,dx}{\frac{1}{3}ab} = \frac{3}{10}a \text{ from } OC;$$

$$\bar{y} = \frac{\frac{1}{2}\int b^2 dx - \frac{2b^2}{\sqrt{a}} x^{\frac{1}{2}}\,dx + \frac{b^2}{a} x\,dx}{\frac{1}{3}ab} = \frac{1}{4}\,b \text{ from } BC,$$

or $\frac{3}{4}$ of OC from OA.

These last two values can be readily determined from the first two by the application of the principle of moments.

Thus the area $OBA = \frac{2}{3}ab$, and area $OBC = \frac{1}{3}ab$, and the sum of the moments of these areas with reference to the axes of x and y must equal the moment of the rectangle $OABC$. Hence

$$\frac{2}{3}ab \times \frac{3}{5}a + \frac{1}{3}ab \times \bar{x} = ab \times \frac{1}{2}a, \quad \text{or} \quad \bar{x} = \frac{3}{10}a;$$

$$\frac{2}{3}ab \times \frac{3}{8}b + \frac{1}{3}ab \times \bar{y} = ab \times \frac{1}{2}b, \quad \text{or} \quad \bar{y} = \frac{3}{4}b.$$

(2) ***Find the centre of mass for the area of a quadrant of a circle in which the density increases directly as the distance from the centre.***

If δ_1 is the surface density at a units distance, the density at any distance ρ

is proportional to $\delta_1\rho$. Putting this in the place of δ in equation (10) we have, if r is constant,

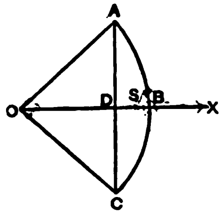
$$\bar{x}=\bar{y}=\frac{\delta_1\int_0^{\frac{\pi}{2}}\int_0^r \rho^3 d\rho\cos\theta d\theta}{\delta_1\int_0^{\frac{\pi}{2}}\int_0^r \rho^2 d\rho d\theta}=\frac{\frac{1}{4}r^4}{\frac{1}{6}\pi r^3}=\frac{3r}{2\pi}.$$

If the density is constant, we have from equation (11)

$$\bar{x}=\bar{y}=\frac{\int_0^{\frac{\pi}{2}}\int_0^r \rho^2 d\rho\cos\theta d\theta}{\frac{1}{4}\pi r^2}=\frac{4r}{3\pi}.$$

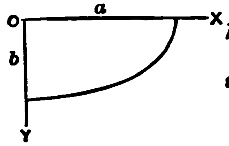
(3) *Find the centre of mass of the area of a homogeneous circular segment.*

Let the origin be at the centre of the circle, and the axis of x the axis of symmetry. Let the chord $AC = c$ and the radius r. Then the equation of the circle is $x^2+y^2=r^2$. Hence $xdx = -ydy$. From equation (9)



$$\bar{x}=\frac{\int_{+\frac{c}{2}}^{-\frac{c}{2}} -y^2dy}{A}=\frac{c^3}{12A}.$$

The centre of mass of a homogeneous circular segment is on the radius drawn to the middle of the arc, at a distance OS from the centre of the circle equal to the cube of the chord divided by twelve times the area of the segment. [See (9), page 24.]



(4) *Find the centre of mass of the area of a homogeneous quadrant of an ellipse.*

The equation of the ellipse referred to its centre and axes is $a^2y^2+b^2x^2=a^2b^2$.

Hence $xdx = -\frac{a^2}{b^2}ydy$, and $y^2=b^2-\frac{b^2}{a^2}x^2$.

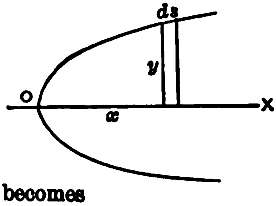
From equation (9) we have

$$\bar{x}=\frac{\int_0^{-b} -\frac{a^2}{b^2}y^2dy}{\frac{1}{4}\pi ab}=\frac{4a}{3\pi};$$

$$\bar{y}=\frac{\frac{1}{2}\int_0^a b^2dx-\frac{b^2}{a^2}x^2dx}{\frac{1}{4}\pi ab}=\frac{4b}{3\pi}.$$

If $a=b$, the ellipse becomes a circle, and the co-ordinates of the centre of mass of a circular quadrant referred to its centre are $\bar{x}=\bar{y}=\frac{4r}{3\pi}$, as in example (2).

[**Centre of Mass of Curved Surfaces.**—If the surface is one of revolution, let the axis of x coincide with the axis of revolution, which is also an axis of symmetry. The surface can be divided by planes perpendicular to the axis into a series of circular rings. Let ds be the length element of the generating curve. The elementary surface generated by its revolution will be $2\pi y ds$. If the thickness of the surface is τ, the elementary volume is $dV = 2\tau\pi y ds$, and equation (8) becomes



$$\bar{x} = \frac{\int 2\tau\pi\delta x y ds}{\int 2\tau\pi\delta y ds} = \frac{\int \tau\delta x y ds}{\int \tau\delta y ds}. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If τ is constant and the surface homogeneous, δ is constant, and $\int 2\pi y ds = A =$ the entire area of the surface, and

$$\bar{x} = \frac{2\pi \int x y ds}{A}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

For curved surfaces in general we have $dV = \tau da$ and equation (8) becomes

$$\bar{x} = \frac{\int \tau\delta x da}{\int \tau\delta da}, \quad \bar{y} = \frac{\int \tau\delta y da}{\int \tau\delta da}, \quad \bar{z} = \frac{\int \tau\delta z da}{\int \tau\delta da}. \quad . \quad . \quad (14)$$

If τ is constant and the surface homogeneous, we have

$$\bar{x} = \frac{\int x da}{A}, \quad \bar{y} = \frac{\int y da}{A}, \quad \bar{z} = \frac{\int z da}{A}. \quad . \quad . \quad . \quad (15)$$

The elementary area

$$da = \frac{dxdy}{\cos\theta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where θ is the angle which the tangent plane to the surface makes with the plane xy, and is given by

$$\cos\theta = \pm \frac{\frac{dL}{dz}}{\sqrt{\frac{dL^2}{dx^2} + \frac{dL^2}{dy^2} + \frac{dL^2}{dz^2}}}, \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where $L = f(x, y, z) = 0$ is the functional equation of the surface.

EXAMPLES.

(1) *Find the centre of mass of one eighth of the surface of a spherical shell of uniform thickness and density.*

The equation of the sphere, if r is the radius, is

$$L = x^2 + y^2 + z^2 - r^2 = 0.$$

Hence $\frac{dL}{dx} = 2x$, $\frac{dL}{dy} = 2y$, $\frac{dL}{dz} = 2z$, and equations (17) and (16) become

$$\cos\theta = \frac{2z}{\sqrt{4x^2+4y^2+4z^2}} = \frac{z}{r}, \quad da = \frac{rdxdy}{z} = \frac{rdxdy}{\sqrt{r^2-x^2-y^2}}.$$

Therefore from equation (15), if we put $r^2 - x^2 = v^2$, since $A = \frac{1}{2}\pi r^2$,

$$\bar{x} = \frac{\int_0^r \int_0^v \frac{rxdxdy}{\sqrt{v^2-y^2}}}{\frac{1}{2}\pi r^2} = \frac{\int_0^r \frac{\pi r}{2} xdx}{\frac{1}{2}\pi r^2} = \frac{1}{2}r.$$

Also

$$\bar{y} = \frac{\int_0^r \int_0^v \frac{rdx \,.\, ydy}{\sqrt{v^2-y^2}}}{\frac{1}{2}\pi r^2} = \frac{\int_0^r r\sqrt{r^2-x^2}dx}{\frac{1}{2}\pi r^2} = \frac{1}{2}r;$$

$$\bar{z} = \frac{\int_0^r \int_0^v rdxdy}{\frac{1}{2}\pi r^2} = \frac{\int_0^r r\sqrt{r^2-x^2}dx}{\frac{1}{2}\pi r^2} = \frac{1}{2}r.$$

If the thickness of the shell varies as the ordinate z, then $\tau = cz$, and from equation (14)

$$\bar{x} = \frac{\int_0^r \int_0^v rxdxdy}{\int_0^r \int_0^v rdxdy} = \frac{4r}{3\pi};$$

$$\bar{y} = \frac{\int_0^r \int_0^v rydxdy}{\int_0^r \int_0^v rdxdy} = \frac{4r}{3\pi};$$

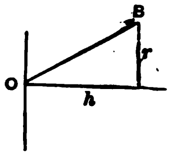
$$\bar{z} = \frac{\int_0^r \int_0^v r(r^2-x^2-y^2)^{\frac{1}{2}}dxdy}{\int_0^r \int_0^v rdxdy} = \frac{2r}{3}.$$

(2) *Find the centre of mass of a thin shell of uniform density and thickness, generated by the revolution of a quadrant of a circle about one radius.*

The equation of the generating curve is $x^2 + y^2 = r^2$, hence $dy = -\frac{xdx}{y}$, $ds = \sqrt{dx^2 + dy^2} = \frac{rdx}{y}$ and $yds = rdx$. Since $A = 2\pi r^2$, we have from equation (18)

$$\bar{x} = \frac{2\pi \int_0^r rxdx}{2\pi r^2} = \frac{r}{2}.$$

(3) *Find the centre of mass of a right conical surface of uniform thickness and density.*



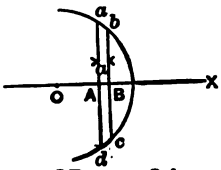
Let the altitude be h and the radius of the base r. Then the equation of the generating line is $y = \frac{r}{h}x$. Hence $dy = \frac{r}{h}dx$, and $ds = \sqrt{dx^2 + dy^2} = \frac{dx}{h}\sqrt{h^2 + r^2}$.

If l is the slant height OB, then $ds = \frac{l}{h}dx$, $yds = \frac{rl}{h^2}xdx$, and $yxds = \frac{rl}{h^2}x^2dx$. The area $A = \pi rl$. Hence from equation (13)

$$\bar{x} = \frac{2\pi\int_0^h \frac{rl}{h^2}x^2dx}{\pi rl} = \frac{2}{3}h.$$

Or the centre of mass of a right conical surface is on the axis at a distance from the vertex of two thirds the altitude (page 25).

(4) *Find the centre of mass of the surface of a spherical segment, zone or hemisphere, of uniform thickness and density.*



The equation of the generating curve is $x^2 + y^2 = r^2$, hence $dy = -\frac{xdx}{y}$ and $ds = \sqrt{dx^2 + dy^2} = \frac{rdx}{y}$.

The area of the surface is then

$$A = \int_{x_1}^{x_2} 2\pi rdx = 2\pi r(x_2 - x_1) = 2\pi ra,$$

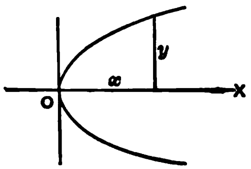
where a is the altitude AB of the segment or zone, and $x_2 = OB$, $x_1 = OA$.

From (13) we have

$$\bar{x} = \frac{2\pi\int_{x_1}^{x_2} rxdx}{2\pi ra} = \frac{x_2 + x_1}{2}.$$

Hence the centre of mass is at the middle of its altitude (page 26).

(5) *Find the centre of mass of the surface of a paraboloid of revolution, of uniform density and thickness.*



We have for the equation of the generating curve $y^2 = 2px$, hence $dy = \frac{pdx}{y}$ and

$$ds = \sqrt{dx^2 + \frac{p^2dx^2}{y^2}} = \frac{dx}{y}\sqrt{2px + p^2},$$

$$yds = dx\sqrt{2px + p^2}.$$

Therefore

$$A = 2\pi\int^x yds = \frac{2\pi}{3p}\sqrt{(2px + p^2)^3}$$

and

$$\bar{x} = \frac{2\pi\int_0^x yxds}{A} = \frac{2\pi\int_0^x xdx\sqrt{2px + p^2}}{A} = \frac{2\pi(3px - p^2)\sqrt{(2px + p^2)^3}}{15p^2A},$$

or

$$\bar{x} = \frac{3x - p}{5}.$$

(6) ***Find the centre of mass of a thin shell of uniform thickness and density formed by the revolution of a semi-cycloid about its base.***

The equation of the generating curve is

$$x = r \text{ versin}^{-1}\frac{y}{r} - (2ry - y^2)^{\frac{1}{2}}.$$

Hence

$$\frac{dx}{y} = \frac{dy}{(2ry - y^2)^{\frac{1}{2}}} = \frac{ds}{(2ry)^{\frac{1}{2}}}$$

$$\bar{x} = \frac{\displaystyle\int_0^{2r} \frac{xydy}{(2r - y)^{\frac{1}{2}}}}{\displaystyle\int_0^{2r} \frac{ydy}{(2r - y)^{\frac{1}{2}}}} = \frac{26r}{15}.$$

[**Centre of Mass of Bodies.**—Let us consider first a solid of revolution, and take the axis of revolution as the axis of x. Take a slice at right angles to x, whose thickness is dx. Take a particle of this slice at a distance r from the axis, and let the plane which passes through x and the particle make the angle θ with the plane of xy. Then the volume of an element is $dV = rd\theta drdx$. If δ is the density, the mass is $\delta rd\theta drdx$.

If the density is symmetrical with respect to the axis of revolution, the centre of mass is on this axis, and we have

$$x = \frac{\int\int\int \delta rxd\theta drdx}{\int\int\int \delta rd\theta drdx}.$$

If we perform the θ integration between $\theta = 0$ and $\theta = 2\pi$, since the symmetry of the body renders δ independent of θ, we have

$$\bar{x} = \frac{\pi\int\int \delta rxdrdx}{\pi\int\int \delta rdrdx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

If the density is uniform throughout a complete slice, we may perform the r integration between $r = 0$ and $r = y$, where y is the ordinate of the generating curve, and we have

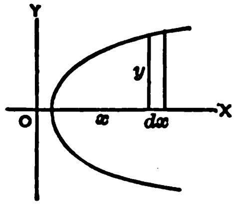
$$\bar{x} = \frac{\pi\int \delta y^2xdx}{\pi\int \delta y^2dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

If δ is uniform, the total volume is

$$V = \pi\int y^2dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

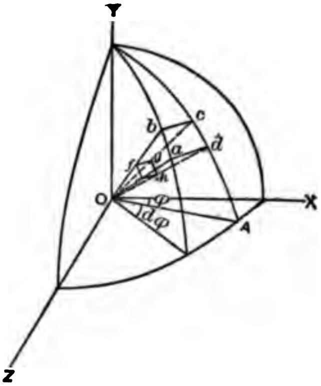
and we have for homogeneous solids of revolution

$$\bar{x} = \frac{\pi\int xy^2dx}{V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$



We see at once from the figure that πy^2dx is the volume of a slice, and the moment of this slice with reference to the axis of y is $\pi y^2dx \times x$. Hence (20) and (21).

For a body in general we have $dV = dxdydz$, and hence equations (8) become



$$\left.\begin{aligned} \bar{x} &= \frac{\int\int\int \delta x dx dy dz}{\int\int\int \delta dx dy dz}; \\ \bar{y} &= \frac{\int\int\int \delta y dx dy dz}{\int\int\int \delta dx dy dz}; \\ \bar{z} &= \frac{\int\int\int \delta z dx dy dz}{\int\int\int \delta dx dy dz}. \end{aligned}\right\} \quad . \quad . \quad (22)$$

If the axis of x is an axis of symmetry, $\bar{x}$ is sufficient.

For polar co-ordinates let $\phi = AOX$, $\theta = dOA$, $\rho = Oh$. Then $hd = d\rho$, $hg = \rho d\theta$, $he = \rho \cos\theta d\phi$, $dV = hd \times hg \times he = \rho^2 d\rho \cos\theta d\theta d\phi$.

Also, $\quad x = \rho\cos\theta\cos\phi, \quad y = \rho\sin\theta, \quad z = \rho\cos\theta\sin\phi.$

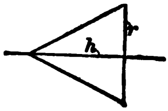
Hence, from equations (3),

$$\left.\begin{aligned} \bar{x} &= \frac{\int\int\int \delta\rho^3 d\rho \cos^2\theta d\theta \cos\phi d\phi}{\int\int\int \delta\rho^2 d\rho \cos\theta d\theta d\phi}; \\ \bar{y} &= \frac{\int\int\int \delta\rho^3 d\rho \cos\theta \sin\theta d\theta d\phi}{\int\int\int \delta\rho^2 d\rho \cos\theta d\theta d\phi}; \\ \bar{z} &= \frac{\int\int\int \delta\rho^3 d\rho \cos^2\theta d\theta \sin\phi d\phi}{\int\int\int \delta\rho^2 d\rho \cos\theta d\theta d\phi}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (23)$$

For a homogeneous body δ disappears in (23) and the denominator becomes the total volume V.

EXAMPLES.

(1) *Find the centre of mass of a right cone of uniform density.*



The equation of the generating line is $y = \frac{r}{h}x$, where h is the altitude and r the radius of the base. The volume is $V = \frac{\pi r^2 h}{3}$. Hence from equation (21)

$$\bar{x} = \frac{\pi \int_0^h \frac{r^2}{h^2} x^3 dx}{\frac{1}{3}\pi r^2 h} = \frac{3}{4}h.$$

That is, the centre of mass is at a distance from the vertex equal to three fourths of the axis. [See (15), page 26.]

(2) *Find the centre of mass of a paraboloid of revolution of uniform density the length of whose axis measured from the vertex is h.*

The equation of the generating curve is $y^2 = \frac{r^2}{h}x$, where r is the radius of the base. The volume is $V = \frac{\pi r^2 h}{2}$. Hence from equation (21)

$$\bar{x} = \frac{\pi \int_0^h \frac{r^2}{h} x^2 dx}{\frac{1}{2}\pi r^2 h} = \frac{2}{3}h.$$

That is, the centre of mass is at a distance from the vertex equal to two thirds of the axis.

(3) *Find the centre of mass of a semi-circular spherical wedge, of uniform density, and radius r.*

From equation (23), integrating between the limits $\rho = 0$, $\rho = r$, and $\theta = +\frac{\pi}{2}$, $\theta = -\frac{\pi}{2}$, we have, since $V = \frac{\phi}{2\pi} \cdot \frac{4}{3}\pi r^3$,

$$\bar{x} = \frac{\frac{r^4}{4} \cdot \frac{\pi}{2} \sin\phi}{\frac{\phi}{2\pi} \cdot \frac{4}{3}\pi r^3} = \frac{3\pi r}{16} \cdot \frac{\sin\phi}{\phi}.$$

If the angle ϕ is small, $\sin\phi = \phi$ and $\bar{x} = \frac{3\pi r}{16}$.

If $\phi = \frac{\pi}{2}$, we have for the *hemisphere* $\bar{x} = \frac{3}{8}r$ (page 28).

(4) *Find the centre of mass of a portion of a spheroid of uniform density, the length of whose axis measured from the vertex is h.*

Let the equation of the generating curve be the ellipse referred to its vertex,

$$y^2 = \frac{b^2}{r^2}(2rx - x^2),$$

where r is the semi-major axis and b is the semi-minor axis.

Then from equation (19)

$$\bar{x} = \frac{\int_0^h (2rx - x^2)x dx}{\int_0^h (2rx - x^2) dx} = \frac{h}{4}\,\frac{8r - 3h}{3r - h}.$$

For a hemispheroid $h = r$ and $\bar{x} = \frac{5}{8}r$ from the vertex.

As b does not enter into these values, they are the same for a spherical segment and for a hemisphere.

For the distance from the centre we have

$$OS = r - \bar{x} = \frac{3}{4}\,\frac{(2r - h)^2}{3r - h},$$

as already found in (18), page 28.

(5) *Find the centre of mass of an octant of a sphere of uniform density.*

From equation (28) we have, since δ disappears and $V = \frac{1}{6}\pi r^3$,

$$\bar{x} = \frac{\int_0^r \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \rho^3 d\rho \cos^2\theta d\theta \cos\phi d\phi}{\frac{1}{6}\pi r^3} = \frac{3}{8}r;$$

$$\bar{y} = \frac{\int_0^r \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \rho^3 d\rho \cos\theta \sin\theta d\theta d\phi}{\frac{1}{6}\pi r^3} = \frac{3}{8}r;$$

$$\bar{z} = \frac{\int_0^r \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \rho^3 d\rho \cos^2\theta d\theta \sin\phi d\phi}{\frac{1}{6}\pi r^3} = \frac{3}{8}r.$$

(6) *Let the density in the preceding example vary as the nth power of the distance from the centre.*

Let $\delta = c\rho^n$. Then from equations (28) we have

$$\bar{x} = \frac{\int_0^r \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \rho^{n+3} d\rho \cos^2\theta d\theta \cos\phi d\phi}{\int_0^r \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \rho^{n+2} d\rho \cos\theta d\theta d\phi} = \frac{n+3}{n+4}\frac{r}{2} = \bar{y} = \bar{z}.$$

(7) *Find the centre of mass of one eighth of the volume of an ellipsoid of uniform density contained within the three principal planes.*

Let the semi-axes of the ellipsoid be a, b, c.

The volume of the ellipsoid is $\frac{4}{3}\pi abc$. The volume of one eighth is therefore $V = \frac{1}{6}\pi abc$.

The equations of the curve on the three principal planes are

$$a^2y^2 + b^2x^2 = a^2b^2, \quad a^2z^2 + c^2x^2 = a^2c^2, \quad b^2z^2 + c^2y^2 = b^2c^2.$$

Therefore we have

$$y = \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}, \quad z = \frac{c}{a}(a^2 - x^2)^{\frac{1}{2}}, \quad z = \frac{c}{b}(b^2 - y^2)^{\frac{1}{2}}.$$

$$x = \frac{a}{b}(b^2 - y^2)^{\frac{1}{2}}, \quad x = \frac{a}{c}(c^2 - z^2)^{\frac{1}{2}}, \quad y = \frac{b}{c}(c^2 - z^2)^{\frac{1}{2}}.$$

The volume of a slice parallel to YZ, of thickness dx, is $\frac{\pi yz}{4}dx$.

" " " " " " " XZ, " " dy, is $\frac{\pi xz}{4}dy$.

" " " " " " " XY, " " dz, is $\frac{\pi xy}{4}dz$.

Hence

$$\bar{x} = \frac{\displaystyle\int \frac{\pi yz}{4} x\,dx}{\frac{1}{6}\pi abc} = \frac{\displaystyle\frac{\pi bc}{4a^2}\int_0^a (a^2 - x^2)x\,dx}{\frac{1}{6}\pi abc} = \frac{\frac{1}{16}\pi bca^2}{\frac{1}{6}\pi abc} = \frac{3}{8}a;$$

$$\bar{y} = \frac{\displaystyle\int \frac{\pi xz}{4} y\,dy}{\frac{1}{6}\pi abc} = \frac{\displaystyle\frac{\pi ac}{4b^2}\int_0^b (b^2 - y^2)y\,dy}{\frac{1}{6}\pi abc} = \frac{\frac{1}{16}\pi acb^2}{\frac{1}{6}\pi abc} = \frac{3}{8}b;$$

$$\bar{z} = \frac{\displaystyle\int \frac{\pi xy}{4} z\,dz}{\frac{1}{6}\pi abc} = \frac{\displaystyle\frac{\pi ab}{4c^2}\int_0^c (c^2 - z^2)z\,dz}{\frac{1}{6}\pi abc} = \frac{\frac{1}{16}\pi abc^2}{\frac{1}{6}\pi abc} = \frac{3}{8}c.$$

CHAPTER IV.

LINE REPRESENTATIVE OF A FORCE. COMPOSITION AND RESOLUTION OF FORCES.

FORCE OF GRAVITATION. ATTRACTION OF A HOMOGENEOUS SHELL OR SPHERE. CENTRE OF GRAVITY. VALUE OF CONSTANT OF GRAVITATION. ASTRONOMICAL UNIT OF MASS. VALUE OF a' FOR PLANETARY MOTION. ATTRACTION OF A CIRCULAR ARC. ATTRACTION OF A STRAIGHT LINE. ATTRACTION OF A CIRCULAR RING. ATTRACTION OF A CIRCULAR DISK. ATTRACTION OF A CYLINDER. ATTRACTION OF A CONE. VALUE OF g ABOVE SEA-LEVEL.

Line Representative of a Force.—We have seen (page 2) that the force on a particle acts in the direction of the acceleration it causes, and that the magnitude of the force is proportional to the acceleration.

Force then has magnitude and direction, and is therefore a vector quantity, and can be represented, like linear acceleration, by a straight line.

$$A \xrightarrow{F = mf} B$$

Thus the length of the line AB represents the magnitude of the force $F = mf$ (page 5). Its point of application is A, and its direction of action is indicated by the arrow and is always the same as that of the acceleration f.

Composition and Resolution of Forces.—The principles, therefore, of pages 35, 43, 49 (Vol. I, *Kinematics*) hold good for forces as well as for displacements, velocities and accelerations, and we can resolve and combine forces and have the "triangle and polygon of forces" as well as the triangle and polygon of displacements, velocities or accelerations.

An important case of the composition of forces is the determination of the attractive force exerted on a particle by an extended body. The attraction on the particle in such case is the resultant of all the attractions exerted upon it by the particles of the body.

Force of Gravitation.—The "law of gravitation" as formulated by Newton asserts that *every particle of matter attracts every other particle with a force which acts in the straight line joining the particles and whose magnitude is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them.*

If then M and m are the masses of two particles and r the distance between them, the mutual force of attraction F is given by

$$F = \kappa \frac{Mm}{r^2}, \qquad \qquad (1)$$

where κ is a constant to be determined by experiment.

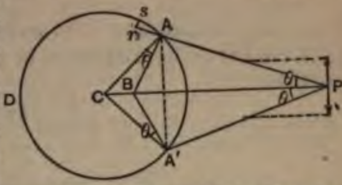
For absolute accuracy and universal generality, as well as for far-reaching consequences, this statement is without parallel in the

history of science. The facts that by means of it the motions of all the bodies of the solar system are explained completely; that their past and future positions can be told; that the existence of Neptune was deduced from the assumption that certain disturbances in the motion of Uranus were due to the attraction of an unknown planet according to this law, all go to prove that the law holds with absolute accuracy, so far as the action upon each other of large masses *separated by distances which are great compared with their linear dimensions is concerned.*

The terms of the enunciation of the law expressly confine it to such cases, since only when the linear dimensions of the attracting bodies are insignificant compared to the distance between them can we consider them as particles and speak of the distance between them.

We shall, however, show in the next Article that if bodies are homogeneous and spherical, this limitation may be removed and the "distance between them" is the distance between their centres.

Attraction of a Homogeneous Shell or Sphere.—Let the circle ADA', with centre at C, represent a uniform thin homogeneous spherical shell whose surface density (page 10) is δ. Suppose a particle at P whose mass is m. Join C and P. Take any point A of the shell and draw CA and AP. Let AP make the angle θ with CP, and draw a line AB through A, making the same angle θ with CA.



Then in the two triangles CAB and CAP we have the side CA and the angle at C common to both, and the angles at A and P equal by construction. These triangles are therefore similar and we have

$$\frac{AB}{AP} = \frac{CA}{CP}.$$

Now let As represent any small elementary area of the spherical surface, and An its projection normal to AB.

Let ω square radians (Vol. I, page 7) denote the conical angle subtended at B by An. Then the area denoted by An is equal to $\overline{AB}^2 . \omega$, and the area denoted by As is equal to $\frac{\overline{AB}^2 . \omega}{\cos\theta}$, since the angle $nAs = BAC = \theta$, and the angle snA is a right angle.

The mass of the elementary area denoted by As is then $\frac{\delta \overline{AB}^2 . \omega}{\cos\theta}$, and the attraction of this mass for the particle of mass m at P is, by Newton's law,

$$k\frac{m . \delta\overline{AB}^2 . \omega}{\overline{AP}^2 \cos\theta},$$

and acts in the line AP.

If we draw AA' perpendicular to CP, we have evidently the same attraction between the equal elementary mass at A' and the particle of mass m at P acting in the line $A'P$.

We can resolve each of these equal forces into a component along the line CP and at right angles to CP at P. Since the angles APC and $A'PC$ are each equal to θ, the two components at right angles to CP at P are equal and opposite and therefore produce no

effect upon P. The resultant attraction of the two elements at A and A' upon the particle of mass m at P acts then in the line CP and is equal to

$$2\kappa \frac{m \,.\, \delta \overline{AB}^2 \,.\, \omega}{\overline{AP}^2 \cos\theta} \cos\theta = 2\kappa \frac{m \,.\, \delta \overline{AB}^2 \,.\, \omega}{\overline{AP}^2},$$

or since $\dfrac{AB}{AP} = \dfrac{CA}{CP}$, the resultant attraction is

$$2\kappa \frac{m \,.\, \delta \overline{CA}^2 \,.\, \omega}{\overline{CP}^2}.$$

But $\overline{CA}^2 \,.\, \omega$ is the area of the elementary area at A or A', and $2\kappa \dfrac{m\delta}{\overline{CP}^2}$ is constant for all pairs of elements A and A'. The total attraction of the shell for the particle of mass m at P acts then in the line CP and is equal to

$$2\kappa \frac{m \,.\, \delta}{\overline{CP}^2} \Sigma\, \overline{CA}^2 \,.\, \omega,$$

where the summation is to be taken for an entire hemisphere. But $\Sigma \overline{CA}^2 .\, \omega$ for a hemisphere is $2\pi CA^2$, and hence the attraction is equal to

$$F = \kappa \frac{4\pi\delta \overline{CA}^2 \,.\, m}{\overline{CP}^2} = \kappa \frac{mM}{\overline{CP}^2},$$

where $M = 4\pi\delta\, \overline{CA}^2$ is the total mass of the spherical shell.

We see, then, that the spherical shell attracts a mass m at any outside point P, just as if its entire mass were condensed at the centre of the shell.

If instead of a homogeneous spherical shell we have a solid homogeneous sphere, we may consider it as composed of an indefinite number of concentric homogeneous spherical shells, each of which attracts the mass at P as if its entire mass were condensed at its centre.

Hence, *the attraction of a homogeneous spherical shell or of a homogeneous sphere upon a particle at any outside point is the same as if the entire mass of the shell or sphere were condensed in a point at the centre.*

We can therefore consider a homogeneous shell or sphere as a particle of equal mass at the centre, so far as its attraction upon an outside particle is concerned.

COR. If the sphere is not homogeneous, but the density of every point at the same distance from the centre is the same, we may still consider the sphere as composed of homogeneous spherical concentric shells, each one of which attracts an outside mass as if its entire mass were condensed at the centre. Hence the same holds true for the sphere.

Centre of Gravity.—When a body attracts and is attracted by all external bodies, whatever their distance and position, as though its mass were condensed in a single point fixed relatively to the body, that point is properly called the **centre of gravity** (see page 18).

A body which has a centre of gravity is said to be **centrobaric** or **barycentric**. In general, bodies are not centrobaric if the law of at-

traction follows Newton's law—that is, if the force is inversely proportional to the square of the distance.

As we have just seen, a homogeneous spherical shell or a homogeneous sphere is centrobaric, and the centre of gravity is at the centre. So also for a non-homogeneous sphere whose density at every point equally distant from the centre is the same. The centre of gravity in each of these cases coincides with the centre of mass (page 16). In general, if a body has a centre of gravity at all, it must always coincide with the centre of mass, because the attraction upon it of an infinitely distant body constitutes a system of parallel particle forces (page 18), and the point of application of the resultant of such a system coincides with the centre of mass.

But while all bodies have a centre of mass, only homogeneous spherical shells and spheres, or spheres whose density at any point equally distant from the centre is the same, possess a centre of gravity.

If, then, the term "centre of gravity" is used to denote centre of mass, as is often done, we should denote the centre of gravity proper by some other term, such as *barycentric point* or *centrobaric point*.

It is, however, much preferable to restrict the term centre of gravity to the definition here given, and use centre of mass as defined (page 16).

COR. If we consider the earth as a sphere whose density is either constant or the same at all points at the same distance from the centre of mass, then, as we have seen, we may consider it as a particle of equal mass at the centre of mass so far as its attraction upon any outside particle is concerned, and the centre of mass is the centre of figure.

The earth is not strictly spherical, but its deviation from sphericity is insignificant. Also, the density is not strictly constant nor strictly the same at all points at the same distance from the centre of mass. But the small distance between the centre of mass of the earth and that point at which in any case of attraction we may consider its mass condensed is insignificant compared to its radius. So far as its attraction for any outside particle is concerned, then, we may consider it as a particle of equal mass at its centre of mass, and the centre of mass as the centre of figure.

Also, since the dimensions of any body with which we experiment at the earth's surface are insignificant compared to the earth's radius, we may consider any such body as a particle.

Value of Constant of Gravitation.—We have seen (page 44) that if M and m are the masses of two particles and r the distance between them, the mutual force of gravitation is given by

$$F = \kappa \frac{mM}{r^2}, \qquad (1)$$

where κ is a constant to be determined by experiment. This constant κ is called the **constant of gravitation.** We are now able to determine it.

Since force is always equal to mass multiplied by the acceleration in the direction of the force (page 5), we have the acceleration of the particle whose mass is m equal to $\frac{F}{m} = \frac{\kappa M}{r^2}$, and the acceleration of the particle whose mass is M equal to $\frac{F}{M} = \frac{\kappa m}{r^2}$. Hence

$$\frac{\text{accel. of } m}{\text{accel. of } M} = \frac{M}{m};$$

that is, *the accelerations are inversely as the masses.* The acceleration, then, of one particle relative to the other considered as fixed is equal to the sum of the accelerations of each, or

$$\text{relative acceleration} = \frac{\kappa(M+m)}{r^2}. \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

We have just seen (page 47, Cor.) that we may treat the earth and any body with which we experiment on its surface as particles, and can take the mass of the earth as condensed at its centre of mass, and the centre of mass as the centre of figure. Equation (1) therefore applies to any body on the earth's surface.

Now when we experiment with a body at the earth's surface, we know that the observed acceleration g due to gravity is the acceleration of the body *relative* to the earth. We have then from (2), if m' is the mass of the earth and b the mass of the body, and if r' is the radius of the earth at the locality for which g is observed,

$$g = \frac{\kappa(m'+b)}{r'^2}.$$

But the mass of the body is insignificant compared to the mass of the earth; or what is the same thing, since the accelerations are inversely as the masses, the acceleration of the earth is insignificant relatively to that of the body. We accordingly find by experiment that g is constant at the same locality *for all bodies*, and neglecting b, this value of g is given by

$$g = \frac{\kappa m'}{r'^2}, \quad \text{or} \quad \kappa = \frac{gr'^2}{m'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

If we substitute this value of κ in equation (1), we have

$$F = \frac{gr'^2}{m'} \cdot \frac{mM}{r^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Equation (4) gives the force of attraction between two particles of mass m and M at a distance r, the mass of the earth being m', its radius r' at the locality where the acceleration of gravity is g. We see that equation (4) is homogeneous, and we have force equal to mass multiplied by acceleration.

If we take mass in pounds and distance in feet and acceleration in ft.-per-sec. per sec., we have F in poundals. If we take mass in grams and distance in centimeters and acceleration in cm.-per-sec. per sec., we have F in dynes (page 5). If we divide out the g, we have F in gravitation units (page 6).

Astronomical Unit of Mass.—*The astronomical unit of mass is that mass which at units distance attracts an equal mass with unit force.*

From equation (4) of the preceding Article, if we take m and M each equal to m_0, and take r equal to one unit of distance $[L]$, and F equal to one unit of force $[F]$, we have

$$[F] = \frac{gr'^2 m_0^2}{m'[L]^2}, \quad \text{or} \quad m_0 = \sqrt{\frac{m'[L]^2[F]}{gr'^2}}. \quad . \quad . \quad . \qquad (1)$$

Equation (1) gives by definition the astronomical unit of mass. We see that it is homogeneous.

If we insert the mean radius of the earth r' in feet, the corresponding value of g in ft.-per-sec. per sec. and the mass of the earth

m' in pounds, we have very nearly, for the astronomical unit of mass,

$$m_0 = 29063 \text{ lbs.}$$

If we insert r' in centimeters, g in cm.-per-sec. per sec. and m' in grams, we have very nearly, for the astronomical unit of mass,

$$m_0 = 3928 \text{ grams.}$$

If we take m and M in equation (4) of the preceding Article in units of astronomical mass, we have

$$F = \frac{gr'^2}{m'} \cdot \frac{mM}{r^2} \cdot m_0^2 = \frac{mM[L]^3}{r^2}[F].$$

This equation we see is homogeneous. If, then, we adopt the astronomical unit of mass instead of the ordinary unit of mass, we have simply the numeric equation

$$F = \frac{mM}{r^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where m and M are the number of astronomical units of mass in the two attracting particles, r the number of units of length in the distance between them, and F the number of units of force in the attraction,

Value of a' for Planetary Motion.—The sun and planets may be considered like the earth, so far as mutual attraction is concerned, as particles of equal mass condensed at the centre of mass. From equation (2), page 48, if we insert the value of $\kappa = \frac{gr'^2}{m'}$ already found, we have then for the relative acceleration of a planet of mass m with reference to the sun of mass M, considered as a fixed point, when the distance is r,

$$\text{relative accel.} = \frac{M+m}{r^2} \cdot \frac{gr'^2}{m'},$$

where m' is the mass of the earth, r' the mean radius of the earth, and g the corresponding acceleration due to gravity at the earth's surface.

At the distance $r = r'$ = radius of the earth the relative acceleration of the planet with reference to the sun regarded as fixed would be then

$$\text{relative accel.} = \left(\frac{M+m}{m'}\right)g.$$

Now in all our equations for planetary motion (Vol. I, *Kinematics*, page 139) we denoted by a' the known acceleration of a point at a known distance r' from a fixed point. If, then, we take this distance r' equal to the earth's radius, we have

$$a' = \frac{M+m}{m'}g. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is the value for a' given on page 144, Vol. I, *Kinematics*, which must be inserted in all our equations for planetary motion (page 139), where M and m are the mass of sun and planet, m' the mass of the earth, and g the acceleration of gravity at the earth's surface.

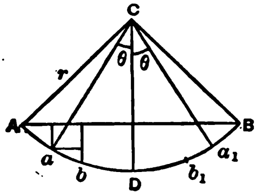
COR. If $M = m' =$ the mass of the earth and m is the mass of a body at the earth's surface, we have

$$a' = \frac{m' + m}{m'} g;$$

or if m is insignificant compared to m',

$$a' = g.$$

Attraction of a Circular Arc.—The attraction of a circular arc ADB of uniform density δ upon a particle at the centre C is *the same as the attraction of a mass equal to the chord with the arc's density concentrated at the middle of the arc at D.*



Take any element of the arc ab, and let it subtend the angle $aCb = \omega$ radians. Then if r is the radius of the circle, $r\omega$ is the length of ab; and if δ is the linear density of the arc, $\delta r\omega$ is the mass of ab. If M is the mass of the particle at C, then $\kappa M \frac{\delta r\omega}{r^2}$ is the attraction of ab for the particle at C, where $\kappa = \frac{gr'^2}{m'}$ (page 48). The attraction of the element $a'b'$ at the same distance on the other side of D will be the same. Each of these can be resolved into components along CD and at right angles to CD at C. The latter components will balance. The sum of the two former is

$$\kappa M . \frac{2\delta r\omega \cos\theta}{r^2}$$

in the direction CD, where θ is the angle aCD.

But $r\omega \cos\theta$ is the projection of ab upon the chord, and if the linear density of the chord is also δ, the mass of the chord projection of ab is $\delta r\omega \cos\theta$. The sum of the attractions of all the pairs of elements will then be

$$A = \kappa M . \frac{\delta . AB}{r^2},$$

or the attraction due to the mass of the chord AB concentrated at D.

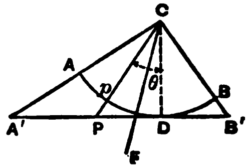
Since $AB = 2r \sin ACD$, we have for the attraction

$$A = \kappa M . \frac{2\delta \sin ACD}{r}.$$

Using the astronomical unit of mass (page 48), we have for the attraction upon a unit mass at C

$$A = \frac{2\delta \sin ACD}{r}.$$

Attraction of a Straight Line.—A limited straight line $A'B'$ of uniform density δ attracts any external particle at C *with the same force and in the same direction as the corresponding arc of a circle AB, of the same density, which has the point C for centre and is tangent to the straight line.*



Let $A'B'$ be the straight line of uniform linear density δ. Draw the arc AB with the centre at C, tangent to the line $A'B'$.

If CpP be drawn cutting the circle at p and the line at P, and we take any element at p and P, subtending the angle ω, then if the angle $PCD = \theta$, we have for the length of the element at p, $Cp \, . \, \omega$, and for the length of the element at P, $\dfrac{CP \, . \, \omega}{\cos\theta}$. The masses of these elements, if the linear density of arc and line is δ, are $\delta \, . \, Cp \, . \, \omega$ and $\dfrac{\delta \, . \, Cp \, . \, \omega}{\cos\theta}$. Their attractions for a mass M at C are

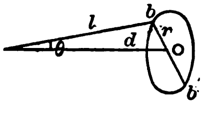
$$\kappa M \frac{\delta \, . \, Cp \, . \, \omega}{Cp^2} = \kappa M \frac{\delta\omega}{Cp} \quad \text{and} \quad \kappa M \frac{\delta \, . \, CP \, . \, \omega}{CP^2 \cos\theta} = \kappa M \frac{\delta\omega}{CP\cos\theta},$$

where $\kappa = \dfrac{gr'^2}{m'}$ (page 48). But $CP \cos\theta = CD = Cp = r$. Hence the attractions of an element at p and P are equal. The arc AB then attracts C as the line $A'B'$ does; and by the preceding Article, using the astronomical unit of mass (page 48), we have for the attraction upon a unit mass at C

$$A = \frac{2\delta \sin \frac{1}{2} A \, CB'}{r}$$

in the direction CF which bisects the angle $A'CB'$.

Attraction of a Circular Ring.—Let r be the radius of the ring, and d the distance of a particle at C of mass M in the perpendicular CO to the plane of the ring through its centre. Take an element of the ring at b which subtends the angle ω. The length of this element is $r\omega$; and if δ is the linear density, the mass in the element is $\delta r\omega$.



The attraction on C is then $\kappa M \dfrac{\delta r\omega}{r^2 + d^2}$, where $\kappa = \dfrac{gr'^2}{m'}$ (page 48).

The attraction of the element at b' at the same distance at the other end of the diameter is the same. Each of these can be resolved into components at right angles to CO at C, which balance, and along CO. The sum of the latter is

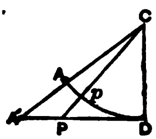
$$\kappa M \frac{2\delta r\omega \cos\theta}{r^2 + d^2},$$

where θ is the angle bCO. But $\cos\theta = \dfrac{d}{r} = \dfrac{d}{\sqrt{r^2 + d^2}}$. Hence we have for each pair of elements the attraction $\kappa M \dfrac{2\delta r\omega d}{(r^2 + d^2)^{\frac{3}{2}}}$.

For the entire ring $\omega = \pi$, and we have, using the astronomical unit of mass (page 48), for the attraction upon a unit mass at C

$$A = \frac{2\pi r d\delta}{(r^2 + d^2)^{\frac{3}{2}}}$$

Attraction of a Circular Disk.—If the line $A'D$ revolves about CD it will generate a circular disk. The arc AD with centre at C and tangent at D to $A'D$ will generate a spherical surface. Then, as we have seen, the attraction of an element at p and P will be equal. If the element at p subtends ω square radians (Vol. I, page 7), its area will be $r^2\omega$, its mass $\delta r^2\omega$, where δ is the surface density, and its attraction



upon a mass M at C will be $\kappa M \frac{\delta r^2 \omega}{r^2} = \kappa M \delta \omega$, where $\kappa \frac{gr'^2}{m'}$ (page 48).

The attraction of the disk whose radius is $A'D = R$ is then the same as the attraction of the spherical surface generated by AD. The number of square radians subtended by the disk of radius R at a distance r from C is $2\pi\left(1 - \frac{r}{\sqrt{r^2 + R^2}}\right)$. The attraction of the disk is then

$$A = \kappa M \,.\, 2\pi\delta\left(1 - \frac{r}{\sqrt{r^2 + R^2}}\right).$$

Using the astronomical unit of mass (page 48), we have for the attraction upon a unit mass at C

$$A = 2\pi\delta\left(1 - \frac{r}{\sqrt{r^2 + R^2}}\right).$$

[Attraction of a Cylinder.]—For the attraction of a cylinder of length l and radius a upon a particle of mass M in the axis at a distance d from its nearest end, let δ be the volume density. Then for the attraction of one of its circular slices of a thickness dx, at a distance x, we have, from the preceding Article,

$$\kappa M \,.\, 2\pi\delta\left[1 - \frac{x}{\sqrt{x^2 + a^2}}\right]dx.$$

If we integrate this between the limits $d + l$ and d, we have

$$A = \kappa M \,.\, 2\pi\delta\left[l - \sqrt{(d + l)^2 + a^2} + \sqrt{d^2 + a^2}\right].$$

If we suppose $d = 0$, so that the particle is on the end surface of the cylinder, we have

$$A = \kappa M \,.\, 2\pi\delta\left[l - \sqrt{l^2 + a^2} + a\right],$$

where $\kappa = \frac{gr'^2}{m'}$ (page 48).

Using the astronomical unit of mass (page 48) we have for the attraction upon a unit mass on the end surface of the cylinder

$$A = 2\pi\delta\left[l - \sqrt{l^2 + a^2} + a\right].$$

[Attraction of a Right Circular Cone.]—For any circular slice we have as before $\kappa M \,.\, 2\pi\delta\left[1 - \frac{x}{\sqrt{x^2 + a^2}}\right]dx$. If θ is the semi-vertical angle of the cone, we have $\cos\theta = \frac{x}{\sqrt{x^2 + a^2}}$. Hence the attraction for a particle of mass M at the vertex is

$$\kappa M \,.\, 2\pi\delta[1 - \cos\theta]\int_0^h dx = \kappa M \,.\, 2\pi\delta(1 - \cos\theta)h,$$

where h is the height of the cone.

Using the astronomical unit of mass (page 48), we have for the attraction for a particle of unit mass at the vertex

$$A = 2\pi\delta(1 - \cos\theta)h.$$

Value of g above Sea-level.—Let r' be the mean radius of the earth, x the height on a mountain above sea-level, and g the acceleration of gravity at sea-level. Then since the acceleration is inversely as the square of the distance, the acceleration at a distance x above sea-level, if we disregard the attraction of the mountain, would be $\frac{r'^2}{(r'+x)^2}g$. To this we must add the acceleration due to the mountain.

Suppose the mountain of uniform density δ and cylindrical in shape, and the particle at the centre of its upper surface. Then the resultant attraction of the mountain for a particle of mass m is, from page 52, if we use the astronomical unit of mass (page 48),

$$A = m \, . \, 2\pi\delta\left[x - \sqrt{x^2+a^2} + a\right],$$

where a is the radius of the cylinder. If we divide the force by m, we obtain the acceleration due to the mountain

$$2\pi\delta\left[x - \sqrt{x^2+a^2} + a\right] = 2\pi\delta\left[x - a\sqrt{1+\frac{x^2}{a^2}} + a\right].$$

If a is so large compared to x that $\frac{x^2}{a^2}$ can be neglected, this reduces to $2\pi\delta x$. If we use the ordinary unit of mass, we have, multiplying by $\kappa = \frac{gr'^2}{m'}$ (page 48), for the acceleration due to the mountain

$$2\pi\delta x \, . \, \frac{gr'^2}{m'}.$$

Let δ' denote the mean density of the earth, so that the mass of the earth is $m' = \frac{4}{3}\pi\delta' r'^3$, then the acceleration due to the mountain is, if we substitute this value of m',

$$\frac{3}{2} \cdot \frac{\delta x}{\delta' r'} g.$$

We have then for the acceleration g' at the height x above sea-level

$$g' = g\left[\frac{r'^2}{(r'+x)^2} + \frac{3\delta x}{2\delta' r'}\right].$$

The mean density of the earth δ' is about $5\frac{1}{2}$ times that of water, and $\frac{\delta}{\delta'}$, from what we know of the density of matter at the earth's surface, may be taken equal to $\frac{1}{2}$. Also we may write

$$\frac{r'^2}{(r'+x)^2} = 1 - \frac{2x}{r'} \text{ approximately.}$$

Hence we have approximately

$$g' = g\left(1 - \frac{2x}{r'} + \frac{3x}{4r'}\right) = g\left(1 - \frac{5x}{4r'}\right),$$

where x is the height above sea-level, r' is the mean radius of the earth, and g the corresponding acceleration due to gravity.

The assumptions made in this investigation are more applicable to elevated table-land than to a mountain. The equation obtained

is the accepted formula for estimating the difference in the value of g at two places so far as dependent on the heights above sea-level.

EXAMPLES.

(1) *If the mass of the earth is* 6.14×10^{27} *grams, the mean radius of the earth* 6.37×10^{8} *cm., and* $g = 981$ *cm.-per-sec. per sec., find the astronomical unit of mass.*

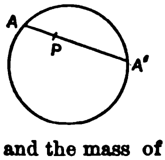
Ans. 3928 grams.

(2) *If the mass of the earth is* 11920×10^{21} *lbs., the mean radius of the earth* 21×10^{6} *ft., and* $g = 32$ *ft.-per-sec. per sec., find the astronomical unit of mass.*

Ans. 29063 lbs.

(3) *Show that the attraction of a thin spherical shell of uniform thickness and density upon a particle inside is zero.*

Ans. Let P be the particle of mass M. Take any point A on the spherical surface. Join AP and produce to A'. If from all points of a small element of the surface at A lines be drawn through P, they will mark off a corresponding element at A'. Both these elements subtend the same conical angle (Vol. I, page 7), ω square radians. The area of the element at A is then $\overline{AP}^2 . \omega$ (Vol. I, page 7), and the area of the element at A' is $\overline{A'P}^2 . \omega$. If δ is the uniform surface density, the mass of the element at A is $m = \delta \overline{AP}^2 . \omega$ and the mass of the element at A' is $m' = \delta \overline{A'P}^2 . \omega$. The attraction of the element at A for a particle of mass M at P is then (page 44)



$$\frac{\kappa M \delta \overline{AP}^2 . \omega}{\overline{AP}^2} = \kappa M \delta \omega$$

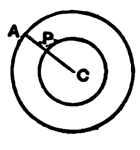
and acts in the line PA. The attraction of the element at A' for the particle of mass M at P is

$$\frac{\kappa M \delta \overline{A'P}^2 . \omega}{\overline{A'P}^2} = \kappa M \delta \omega$$

and acts in the line PA'. The resultant attraction upon the particle at P of the pair of elements at A and A' is then zero. The whole shell consists of such pairs of elements. Hence the resultant attraction of the shell on a particle at P is zero.

(4) *Show that the attraction of a homogeneous sphere on a particle within it is directly proportional to its distance from the centre.*

Ans. Let P be a particle of mass M situated within a homogeneous sphere at any distance PC from the centre C. Then from the preceding example we know that the attraction upon the particle at P due to the shell outside of the sphere whose radius is PC is zero. The attraction upon the particle of mass M at P is then due to the attraction of the sphere whose radius is PC. The volume of the sphere is $\frac{4}{3}\pi \overline{PC}^3$. If δ is the uniform density, the mass of this sphere is $\frac{4}{3}\delta\pi\overline{PC}^3$. Its attraction for a particle of mass M at P is (page 46) the same as if the entire mass of the sphere were condensed at the centre,



or (page 44) $\kappa M \dfrac{\frac{4}{3}\delta\pi\overline{PC}^3}{\overline{PC}^2} = \kappa M . \frac{4}{3}\delta\pi . \overline{PC}.$

The attraction is therefore directly proportional to the distance PC of the particle from the centre.

(5) *Assuming the earth to be a homogeneous sphere, compare its attraction on a given mass at a distance from its centre equal to one half its radius, with the attraction when the given mass is at a distance equal to twice the radius.*

Ans. 2 to 1.

(6) *Find in dynes the attraction of two homogeneous spheres, each of* 100 *kilograms mass, with their centres* 1 *metre apart.*

Ans. 0.0648 dynes nearly.

(7) *How far would a body fall toward the earth in one second from a point at a distance from the earth's surface equal to the radius of the earth?*

Ans. The acceleration is inversely as the square of the distance. We have then $g' : g :: r^2 : 4r^2$, or $g' = \frac{1}{4}g$. That is, the acceleration is one fourth of the acceleration at the surface.

The distance is then $s = \frac{1}{2}g't^2$, or, taking $g = 32$ ft.-per-sec. per sec. and $t = 1$, $s = 4$ ft.

(8) *The moon's mass is* 136×10^{21} *lbs.; the moon's radius,* 5.70×10^6 *ft.; the mass of the earth,* 11920×10^{21} *lbs.; the radius of the earth,* 21×10^6 *ft. Find how far a stone at the moon's surface would fall in a second, the attraction of the earth being neglected.*

Ans. If M is the mass of the moon and m that of the stone, the force of attraction, if r is the radius of the moon, is, from equation (4), page 48,

$$F = \frac{gr'^2}{m'} \frac{mM}{r^2}.$$

The acceleration of the stone is then

$$g' = \frac{F}{m} = \frac{gr'^2}{m'} \cdot \frac{M}{r^2} = \frac{32 \times 21^2 \times 10^{12} \times 136 \times 10^{21}}{11920 \times 10^{21} \times (5.7)^2 \times 10^{12}} = 5 \text{ ft.-per-sec. per sec.}$$

The distance then is $\frac{1}{2}g't^2$, or, taking $t = 1$ sec., $s = 2.5$ ft.

(9) *Suppose the earth to contract until its diameter is* 6000 *miles, what would be the effect on the weight of an inhabitant? The diameter of the earth to be taken at* 8000 *miles.*

Ans. Increased in the ratio of 16 to 9.

(10) *If the mass of the sun is* 300,000 *times the mass of the earth, and its radius is* 100 *times the radius of the earth, find the attraction at the surface of the sun of a mass which at the surface of the earth is attracted by the force of one pound weight.*

Ans. $30g$ poundals, or the attraction of the earth for 30 lbs.

(11) *The diameter of Jupiter is* 10 *times that of the earth, and its mass* 300 *times. By how much per cent of his former weight would the weight of a man be increased by being removed to the surface of Jupiter?*

Ans. By 200 per cent. He would weigh by a spring-balance three times as much as before. The same number of standard pounds would, however, balance him in a lever-balance. The standard pound at Jupiter would be attracted by a force three times as great as the earth's attraction here. The lever-balance weight which gives his mass is unchanged.

(12) *If the intensity of gravity at the surface of Jupiter is about 2.6 times as great as at the surface of the earth, find approximately the time which a body would take in falling from a height of 167 ft. to the surface of Jupiter.*

Ans. 2 sec.

(13) *Find the intensity of the earth's attraction at the distance of the moon, taking 32 ft.-per-sec. per sec. as its value at the surface of the earth. The diameter of the moon's orbit is 480,000 miles, the diameter of the earth 8000 miles.*

Ans. 0.0089 ft.-per-sec. per sec.

[(14)] *Two particles of mass M and m are placed a distance s apart. Find the time it would take them to come together by reason of their mutual attraction, if uninfluenced by any external force.*

Ans. The acceleration of one particle with reference to the other is (page 48)

$$\frac{d^2x}{dt^2} = -\kappa\frac{(M+m)}{x^2}.$$

Integrating (Vol. I, page 102), we have

$$t = \left[\frac{s}{2\kappa(M+m)}\right]^{\frac{1}{2}} \times \left[(sx - x^2)^{\frac{1}{2}} + s\cos^{-1}\left(\frac{x}{s}\right)^{\frac{1}{2}}\right].$$

When $x = s$, $t = 0$; when $x = 0$, we have

$$t = \frac{1}{2}\pi s\left[\frac{s}{2\kappa(M+m)}\right]^{\frac{1}{2}}.$$

If the particles are spheres of density δ and radii R and r, and the density of the earth is δ', we have (page 48)

$$\kappa = \frac{gr'^2}{m'} = \frac{g}{\frac{4}{3}\pi r'\delta'}, \quad M = \frac{4}{3}\pi R^3\delta, \quad m = \frac{4}{3}\pi r^3\delta,$$

and

$$t = \frac{1}{2}\pi s\left[\frac{sr'\delta'}{2\delta g(R^3 + r^3)}\right]^{\frac{1}{2}}.$$

If the spheres are of the same density as the earth, $\delta = \delta'$ and

$$t = \frac{1}{2}\pi s\left[\frac{sr'}{2g(R^3 + r^3)}\right]^{\frac{1}{2}}.$$

The last equation, then, gives the time of coming together of two spheres of radii R and r, of same density as the earth, if considered as *concentrated at their centres*. If the spheres are equal,

$$t = \frac{1}{4}\pi s\left(\frac{sr'}{gr^3}\right)^{\frac{1}{2}}.$$

If, for instance, $s = 1$ ft., $g = 32\frac{1}{6}$ ft.-per-sec. per sec., $r = \frac{1}{4}$ ft., $r' = 20,850,000$ ft.,

$$t = 1788 \text{ sec., or } 29.8 \text{ minutes nearly.}$$

DYNAMICS.

PART I. STATICS.

CHAPTER I.

STATICS—CONCURRING FORCES.

A

FORCES IN EQUILIBRIUM. STATICS. LINE REPRESENTATIVE OF A FORCE. COMPOSITION AND RESOLUTION OF FORCES. SIGN OF COMPONENTS OF A FORCE. CONCURRING FORCES. STATIC, MOLAR AND DYNAMIC EQUILIBRIUM. COMPOSITION AND RESOLUTION OF CO-PLANAR FORCES. CONCURRING FORCES NOT IN THE SAME PLANE. CONDITIONS OF EQUILIBRIUM FOR CONCURRING FORCES.

Forces in Equilibrium.—When all the forces acting upon a particle mutually balance, so that the particle moves as if no force acted upon it, the forces are said to be in **equilibrium**. In such case the particle is either at rest or moves with uniform speed in a straight line (page 2).

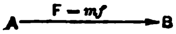
Statics.—That portion of Dynamics which treats of those principles which are necessary for the discussion of forces and bodies in equilibrium, and generally of forces without reference to the change of motion caused by them, is called **Statics**. That portion which treats of forces with reference to the change of motion caused by them is called **Kinetics**.

[Many writers employ the term Dynamics in the sense in which we have used Kinetics, and use the term Mechanics for what we have called Dynamics. They thus have Mechanics divided into Statics and Dynamics, instead of Dynamics divided into Statics and Kinetics.]

Line Representative of a Force.—We have seen (page 2) that the force on a particle acts in the direction of the acceleration it causes, and that the magnitude of the force is proportional to the magnitude of the acceleration.

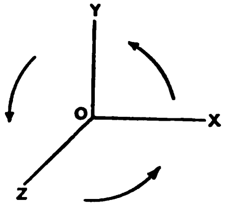
Force, then, has magnitude and direction, and is therefore a vector quantity, and can be represented, like linear acceleration, by a straight line.

Thus the length of the line AB represents the magnitude of the force $F = mf$ (page 5). Its point of application is A, and its direction of action is indicated by the arrow and is always the same as that of the linear acceleration f.



Composition and Resolution of Forces.—The principles, therefore, of pages 35, 43, 49, (Vol. I, Kinematics) hold good for forces also, and we can resolve and combine forces and have the "triangle and polygon of forces" as well as the triangle and polygon of displacements, velocities or accelerations.

We have also the same rule for the signs of the horizontal and vertical components F_x, F_y, F_z of a force as for the corresponding components f_x, f_y, f_z of its acceleration. Thus (+) signifies in the directions O_x, O_y, O_z, and (−) in the opposite directions.



If polar co-ordinates are used, the component force along the radius vector is (+) when it acts away from the pole, (−) when it acts towards the pole.

Evidently, then, we must measure angles in the plane XY, from OX around towards OY; in the plane YZ, from OY around towards OZ; in the plane ZX, from OZ around towards OX.

Concurring Forces, etc.—Forces which act at the same point are called **concurring** forces. Forces acting at different points are **non-concurring**. Forces acting in the same direction in the same line may be called **conspiring** forces; when they act in opposite directions in the same line or in parallel lines they are **opposite forces**; when in the same or opposite direction in parallel lines they are **parallel** forces. Forces whose line representatives lie in the same plane are **co-planar**. Two equal and opposite forces applied at the same point mutually balance, so that the point moves as if no force were applied. (Compare Vol. I, Kinematics, page 178.)

Static Equilibrium.—When all the forces acting upon every particle of a rigid body mutually balance, so that *every particle* of the body moves as if no force acted upon it, the body is said to be in **static** or **molecular** equilibrium. All points of the body in such case are either at rest or they all move with the same uniform speed in parallel straight lines, and the body has a uniform motion of translation (Vol. I, Kinematics, page 91).

The motion of a body is then the same as that of any one of its points, and the body, whatever its size, may be treated as a particle so far as its motion is concerned, and represented by a point.

All the forces acting upon the body itself may then be considered and treated as a system of concurring forces in equilibrium, and all the forces acting upon any one particle of the body also constitute a system of concurring forces in equilibrium.

Molar Equilibrium.—When the centre of mass *only* of a rigid body moves as if no force acted upon it, that is, is either at rest or moves with uniform speed in a straight line, we have equilibrium of the body as a whole, or **molar equilibrium**, as distinguished from molecular or static equilibrium as just defined.

Now the centre of mass of a rigid body always moves as if the mass of the body were condensed into a particle of equal mass at the centre of mass, and all the forces acting upon the entire body were transferred to this particle without change in magnitude and direction (page 18).

When there is molar equilibrium, then, all the forces acting upon the body if appliedat a point would constitute a system of concurring forces in equilibrium. Also all the forces acting upon any particle at the centre of mass of the body constitute a system of concurring forces in equilibrium. But all the forces acting upon any particle *not* at the centre of mass are not in equilibrium, and we have rotation of the body about the centre of mass.

So far as translation of the body alone is concerned, however, we may consider it as a particle of equal mass at the centre of mass, acted upon by a system of concurring forces in equilibrium.

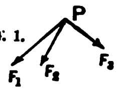
Dynamic or Kinetic Equilibrium.—When *one point only* of a rigid body *not* at the centre of mass moves as though no force acted upon it, the body is said to be in **dynamic** or **kinetic equilibrium** about that point.

In such case all the forces acting at this one point constitute a systemof concurring forces in equilibrium. But the forces acting at any other point do not constitute a system of forces in equilibrium, and we have instantaneous rotation about this point.

Composition and Resolution of Co-planar Forces.—Let the forces F_1, F_2, F_3, etc., be all in the same plane and act either at a common point, P (Fig. 1), or at different points, A, B, C (Fig. 2), of a rigid body.

In either case, lay off the forces so as to obtain the *force polygon* $A\ F_1\ F_2\ F_3$ (Fig. 3). Then the line AF_3 necessary to close this force polygon, taken as acting the other way round, gives the direction and magnitude of the resultant F_r in the plane of the forces (pages 35, 36, Vol. I, Kinematics).

FIG. 1.



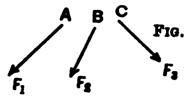


FIG. 2.

If the forces are concurring, or all act at the same point P, Fig. 1, the resultant F_r must act at this point also, in the plane of the forces.

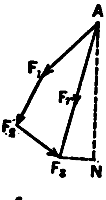


FIG. 3.

If the forces are non-concurring, or act at different points A, B, C, D, Fig. 2, the magnitude and direction of the resultant F_r will still be given by AF_3 in the force polygon, Fig. 3, but its *position* in the plane of the forces is as yet unknown.

COR. 1. If the forces are all parallel, the force polygon Fig. 3 becomes a straight line, and the resultant F_r is equal to the algebraic sum of the forces, or $F_r = \Sigma F$.

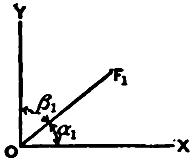
COR. 2. The component AN or NF_3 of the resultant F_r, Fig. 3, in any direction is equal to the algebraic sum of the components of the forces in that direction.

COR. 3. Any number of forces acting upon the same point, whether in the same plane or not, can be reduced to a single resultant force. For the resultant of any two is a force in their plane. This resultant can then be combined with another force, and so on.

COR. 4. If the algebraic sums of the components of the forces in any two directions, as AN and NF_3, are zero, the points A and F_3 in the force polygon Fig. 3 conicide, and the resultant F_r is zero. The forces are then in equilibrium.

Analytical Determination of the Resultant for Concurring Coplanar Forces.—We have evidently the same expressions for the magnitude and direction of the resultant for concurring forces

as for concurring accelerations (page 50, Vol. I, Kinematics).



Thus let any number of co-planar forces, F_1 F_2, etc., all act at the same point O. Take this point as the origin and draw the rectangular axes OX, OY in the plane of the forces. Let F_1 make the angle α_1 with OX, and β_1 with OY; let F_2 make the angle α_2 with OX, and β_2 with OY; and so on.

Denote the algebraic sum of the horizontal components of all the forces by F_x, and the algebraic sum of the vertical components of all the forces by F_y. Then

$$\left.\begin{aligned} F_x &= \Sigma F\cos\alpha = F_1\cos\alpha_1 + F_2\cos\alpha_2 + F_3\cos\alpha_3 + \text{etc.};\\ F_y &= \Sigma F\cos\beta = F_1\cos\beta_1 + F_2\cos\beta_2 + F_3\cos\beta_3 + \text{etc.}\end{aligned}\right\} \quad (1)$$

If F_r is the resultant and a, b the angles which it makes with the axes of x and y respectively, we have for the horizontal and vertical components of F_r (Corollary 2, page 59).

$$\left.\begin{aligned} F_r\cos a &= F_x;\\ F_r\cos b &= F_y.\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Hence

$$\left.\begin{aligned} \cos a &= \frac{F_x}{F_r};\\ \cos b &= \frac{F_y}{F_r}.\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

Squaring and adding, since $\cos b = \sin a$, and $\cos^2 a + \cos^2 b = 1$,

$$F_r = \sqrt{F_x^2 + F_y^2}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

The equation of the line of direction of the resultant, when all the forces act at the origin, is

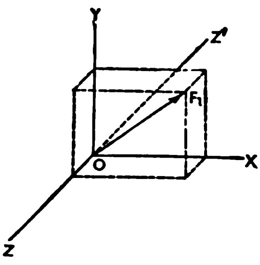
$$y = \frac{F_y}{F_x}x. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

If the co-ordinates of the point at which the forces act are x' and y', the equation of the line of direction of the resultant is in general

$$y - y' = \frac{F_y}{F_x}(x - x'). \quad . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

Equations (1) give the values of F_x and F_y, by which we obtain a, b and F_r from (3) and (4).

The algebraic sums in (1) are found by taking components acting towards the right or upwards as positive, towards the left or downwards as negative (page 58).



Analytical Expression for the Magnitude and Direction of the Resultant of Any Number of Concurring Forces not in the Same Plane.—Let F_1, F_2, F_3, etc., be any number of forces all acting at the same point O. Take this point as the origin for three rectangular axes OX, OY, OZ. Let F_1 make the angles α_1, β_1, γ_1 with these axes respectively, and F_2 make the angles α_2, β_2, γ_2, and so on.

Denote the algebraic sum of the components of all the forces along OX by F_x; along OY by F_y; along OZ by F_z. Then

$$\left.\begin{aligned} F_x &= \Sigma F\cos\alpha = F_1\cos\alpha_1 + F_2\cos\alpha_2 + F_3\cos\alpha_3 + \text{etc.};\\ F_y &= \Sigma F\cos\beta = F_1\cos\beta_1 + F_2\cos\beta_2 + F_3\cos\beta_3 + \text{etc.};\\ F_z &= \Sigma F\cos\gamma = F_1\cos\gamma_1 + F_2\cos\gamma_2 + F_3\cos\gamma_3 + \text{etc.} \end{aligned}\right\} \quad (1)$$

If F_r is the resultant and a, b, c the angles which it makes with the axes of x, y and z respectively, we have

$$\left.\begin{aligned} F_r\cos a &= F_x;\\ F_r\cos b &= F_y;\\ F_r\cos c &= F_z. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence

$$\left.\begin{aligned} \cos a &= \frac{F_x}{F_y};\\ \cos b &= \frac{F_y}{F_r};\\ \cos c &= \frac{F_z}{F_r}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Squaring and adding, since $\cos^2 a + \cos^2 b + \cos^2 c = 1$, we have

$$F_r = \sqrt{F_x^2 + F_y^2 + F_z^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The equations of the projection of the resultant upon the planes of ZX, YX and YZ are

$$x = \frac{F_x}{F_z}z, \quad y = \frac{F_y}{F_x}x, \quad z = \frac{F_z}{F_y}y.$$

Hence from (3) we have for the equation of the line of direction of the resultant, when all the forces act at the origin,

$$\frac{x}{\cos a} = \frac{y}{\cos b} = \frac{z}{\cos c}, \quad \text{or} \quad \frac{x}{F_x} = \frac{y}{F_y} = \frac{z}{F_z}. \quad . \quad . \quad . \quad (5)$$

If the coördinates of the point at which the forces act are x', y', z', we have for the equation of the line of direction of the resultant in general

$$\frac{x-x'}{\cos a} = \frac{y-y'}{\cos b} = \frac{z-z'}{\cos c}, \quad \text{or} \quad \frac{x-x'}{F_x} = \frac{y-y'}{F_y} = \frac{z-z'}{F_z}. \quad (6)$$

When z and F_z equal zero, these equations reduce to the equations of the preceding Article for co-planar forces.

The algebraic sums in (1) are found by taking components acting towards the right along OX, or upwards along OY, or in the direction OZ as positive. The opposite directions are negative.

Conditions of Equilibrium for Concurring Forces.—A point is in equilibrium when its acceleration is zero. In order that the acceleration may be zero, the resultant force acting upon the point must be zero. Hence, *the vanishing of the resultant is the necessary and sufficient condition for equilibrium of any number of concurring forces.*

We have then, in general, the algebraic conditions

$$F_x = \Sigma F\cos\alpha = 0, \quad F_y = \Sigma F\cos\beta = 0, \quad F_z = \Sigma F\cos\gamma = 0.$$

That is, the algebraic sum of the components of the forces in each of any three rectangular directions must be zero. This is equivalent to saying that all the forces acting upon the point reduce to two forces equal in magnitude and opposite in direction.

It is also evident that if any number of forces acting upon a point are in equilibrium, *any one of the forces must be equal and opposite to the resultant of all the others.*

Conditions for Equilibrium for Concurring Forces in Special Cases.—We obtain then the following obvious results from the condition for equilibrium of concurring forces, which will be found useful in special cases:

(1) If two concurring forces are in equilibrium, they must be equal in magnitude and opposite in direction.

(2) If three concurring forces are in equilibrium, they must all act in the same plane. For the resultant of any two must act in their plane and be equal and opposite to the third.

(3) If three concurring forces are represented in magnitude and direction by the sides of a triangle taken the same way round, the resultant is zero and the forces are in equilibrium.

(4) Hence, if three concurring forces are in equilibrium, each one is proportional to the sine of the angle between the other two.

(5) If three concurring forces are in equilibrium and their directions are represented by the sides of a triangle taken the same way round, their magnitudes will also be represented by the sides of that triangle, and *vice versa.*

(6) If any number of concurring co-planar forces are represented in magnitude and direction by the sides of a plane closed polygon taken the same way round, they are in equilibrium. If their magnitudes are given by the sides of the polygon, their directions are also given by the directions of the sides.

But if the directions only of the forces are given by the sides of the plane polygon, it does not follow that the sides of this polygon represent the magnitudes, because any number of plane polygons with parallel sides may be drawn, the magnitudes of the sides varying.

(7) If three concurring forces in different planes are represented by the three edges of a parallelopipedon, the diagonal taken the opposite way round will represent the resultant in direction and magnitude. This is called the *parallelopipedon of forces.*

EXAMPLES.

(1) *Find the resultant of forces of* 7, 1, 1, 3 *units, represented by lines drawn from one angle of a regular pentagon towards the other angles taken in order.*

Ans. $\sqrt{74}$ units.

(2) *P and Q are two component forces at right angles, whose resultant is R. S is the resultant of R and P. If $Q = 2P$ what is S?*

Ans. $S = 2P\sqrt{2}$.

(3) *Component forces P, Q, R are represented in direction by the sides of an equilateral triangle taken the same way round. Find the magnitude of the resultant.*

Ans. $\sqrt{P^2 + Q^2 + R^2 - QR - PR - PQ}$.

(4) *Three component forces are represented by lines drawn from the vertices of a triangle to the middle points of the opposite sides. Show that the resultant is zero.*

(5) *Three component forces are represented by lines drawn from the vertices A, B, C of a triangle to the middle points of the opposite sides, and have magnitudes equal to the cosines of the angles at A, B and C respectively. Find the resultant.*

Ans. $\sqrt{1 - 8 \cos A \cos B \cos C}$ units of force.

(6) *The centre of the circumscribed circle of a triangle ABC is O, and the intersection of the perpendiculars from angular points on opposite sides is P. Prove that the resultant of forces represented in magnitude and direction by OA, OB, OC will be represented by OP.*

(7) *Three forces are represented by the sides AB, AC, BC of a triangle. Show that the resultant has the direction AC and is represented in magnitude by $2AC$.*

(8) *$ABCD$ is a parallelogram. From AB, AE is cut off equal to one third AB. Prove that the resultant of forces represented by AC and $2AD$ is equal to three times the resultant of forces represented by AD and AE.*

(9) *Four forces of 24, 10, 16, 16 dynes act on a particle, the angle between the first and second being 30°, between the second and third 90°, and between the third and fourth 120°. Calculate the resultant.*

Ans. 17.4 dynes.

(10) *A weight of 10 tons is hanging by a chain 20 feet long. Find how much the tension in the chain is increased by the weight being pulled out by a horizontal force to a distance of 12 feet from the vertical.*

Ans. By 2.5 tons.

(11) *A weight of 4 pounds is suspended by a string, and is acted upon by a horizontal force. If in the position of equilibrium the tension of the string is 5 pounds, what is the horizontal force?*

Ans. 3 lbs.

(12) *A mass of 10 lbs. is supported by strings of lengths 3 and 4 feet attached to two points in the ceiling 5 feet apart. What is the tension of each string?*

Ans. 8 lbs. and 6 lbs.

(13) *A particle is acted on by a force whose magnitude is unknown, but whose direction makes an angle of 60° with the horizon. The horizontal component of the force is 1.35 dynes. Determine the total force and its vertical component.*

Ans. 2.7 dynes and 2.34 dynes.

(14) *Three forces proportional to 1, 2, 3, act on a point. The angle between the first and second is 60°, between the second and third 30°. Find the angle which the resultant makes with the first.*

Ans. About 67°.

(15) *Three cords are tied together at a point. One is pulled in a northerly direction with a force of 6 pounds, and another in an easterly direction with a force of 8 pounds. With what force must the third be pulled in order to keep the whole at rest?*

Ans. 10 pounds, at an angle with the horizon whose tang $= \frac{3}{4}$.

(16) *If P and Q are two concurring forces and the angle made by their directions is θ, find the magnitude of the resultant R when $\theta = 0$ and $\theta = \pi$.*

Ans. $(P + Q)$ and $(P - Q)$.

(17) *Find R when $P = Q$ and $\theta = 60°, 135°,$ and $120°$.*

Ans. $R = P\sqrt{3}$; $R = P\sqrt{2 - \sqrt{2}}$; $R = P$.

(18) *If three concurring forces 3, 4 and 5 are in equilibrium, find the angle between the first two.*

Ans. 90°.

(19) *If $P = 6$, $Q = 11$, units, and the angle between P and Q is 30°, find the resultant R, and the angle between P and R and that between Q and R.*

Ans. $R = 16.47$ units; 19° 30′; 10° 30′.

(20) *A cord is tied round a pin at the fixed point A, and its two ends are drawn in different directions by the forces P and Q. If the pressure on the pin is $\frac{P+Q}{2}$, find the angle θ between the forces.*

Ans. $\cos\theta = \dfrac{2PQ - 3(P^2 + Q^2)}{8PQ}$.

(21) *A cord whose length is 2l is tied to the points A and B in the same horizontal line, whose distance is 2a. A smooth ring upon the cord sustains a weight W. Find the tension in the cord.*

Ans. $T = \dfrac{W}{2\sqrt{1 - \frac{a^2}{l^2}}}$.

(22) *Given the four concurring forces $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 4$, and the angles $F_1F_2 = 90°$, $F_2F_4 = 90°$, and $F_1F_3 = 60°$. Find the magnitude of the resultant and its inclination to F_1.*

Ans. $R = 6.889$; 102° 16′.

(23) *Two rafters making an angle of 120° support 112 lbs. at the apex. Find the compressive force on each rafter.*

Ans. 112 lbs. compression.

(24) *Resolve a force of 120 lbs. into two rectangular components, (a) of which one is 75 lbs.; (b) one of which makes an angle of 34° 7′ 3″ with the resultant.*

Ans. (a) 93.65 lbs. making an angle of 38° 40′ 56″.25 with resultant.
(b) 99.343 lbs. adjacent to the given angle and 67.306 lbs.

(25) *The mutually rectangular forces of 35, 67 and 98 lbs. act on a point. Determine the magnitude and direction of the resultant.*

Ans. 123.766 lbs. making angles of 73° 34′ 24″, 57° 13′ 30″, 37° 38′ 42″ with the forces respectively.

(26) *A force of 550 lbs. acts on a point. Resolve it in three rectangular directions, (a) when two of the components are 100 and 230 lbs.; (b) one of the components is 120 lbs. and the given force makes with one of the other two components the angle 15° 6′ 14″; (c) the given force makes with two of the components the angles 87° 13′ 12″ and 54° 17′ 8″.*

Ans. (a) 489.49 lbs., angles 79° 31′ 27″, 65° 16′ 49″ and 27° 7′ 43″. (b) 120 lbs., angle 77° 23′ 51″; 531.02 lbs., angle 15° 6′ 14″; 78.2 lbs., angle 81° 49′ 32″, with resultant. (c) 445.7 lbs., 321.06 lbs., 26.676 lbs.

(27) *A force in space makes with the three co-ordinate axes the angles α, β, γ. Show that* (page 12, Vol. I, *Kinematics*)

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1;$$
$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1;$$
$$\cos(\alpha+\beta)\cos(\alpha-\beta) + \cos^2\gamma = 0.$$

(28) *Two forces acting on a point make the angle ϵ, and make with the co-ordinate axes the angles α_1, β_1, γ_1, and α_2, β_2, γ_2. Show that*

$$\cos\epsilon = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2.$$

(29) *Three forces P, Q, R, acting on a point O, are inclined at angles α, β, γ to a given line passing through O. Find the magnitude and direction of the resultant.*

Ans. If θ is the inclination of the resultant to the given line,

$$\tan\theta = \frac{P\sin\alpha + Q\sin\beta + R\sin\gamma}{P\cos\alpha + Q\cos\beta + R\cos\gamma},$$

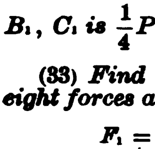
and the resultant is the square root of

$$P^2 + Q^2 + R^2 + 2QR\cos(\beta-\gamma) + 2RP\cos(\gamma-\alpha) + 2PQ\cos(\alpha-\beta).$$

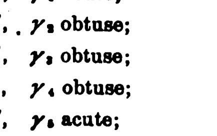
(30) *Three forces, each equal to P, act at a point O in directions OA, OB, OC; the angle AOC being a right angle, and the line OB bisecting the angle AOC. Find the magnitude of the resultant.*

Ans. $P(1+\sqrt{2})$ making an angle of 45° with OA.

(31) *A force P is applied at the hinge A of the knee-joint BAC, making the angle α with AB and AC. Show that the pressure at C and B is $\frac{1}{2}P\tan\alpha$, and that if $P = 50$ lbs. and $\alpha = 15°, 35°, 65°, 85°, 90°$, the pressure is* 6.7, 17.5, 53.6, 285.75 *lbs. and* ∞.



(32) *A force P is applied to the compound knee-joint shown in the accompanying figure. Show that the pressure exerted at B, C and B_1, C_1 is $\frac{1}{4}P\tan\alpha\tan\beta$.*



(33) *Find the resultant for a system of eight forces acting upon a point, given as follows:*

$F_1 = 75$ lbs.;	$\alpha_1 = 63°\,27'$,	$\beta_1 = 48°\,36'$,	γ_1 acute;
$F_2 = 80$ lbs.;	$\alpha_2 = 153°\,44'$,	$\beta_2 = 67°\,13'$,	γ_2 obtuse;
$F_3 = 95$ lbs.;	$\alpha_3 = 76°\,14'$,	$\beta_3 = 147°\,12'$,	γ_3 obtuse;
$F_4 = 185$ lbs.;	$\alpha_4 = 115°\,7'$,	$\beta_4 = 137°\,9'$,	γ_4 obtuse;
$F_5 = 670$ lbs.;	$\alpha_5 = 76°\,3'$,	$\beta_5 = 35°\,3'$,	γ_5 acute;
$F_6 = 37$ lbs.;	$\alpha_6 = 145°\,7'$,	$\beta_6 = 78°\,3'$,	γ_6 acute;
$F_7 = 95$ lbs.;	$\alpha_7 = 62°\,10'$,	$\beta_7 = 149°\,8'$,	γ_7 acute;
$F_8 = 140$ lbs.;	$\alpha_8 = 123°\,58'$,	$\beta_8 = 127°\,56'$,	γ_8 obtuse.

Ans. The angles γ can be found (page 12, Vol. I, *Kinematics*) from

$$\cos(\alpha+\beta)\cos(\alpha-\beta) + \cos^2\gamma = 0.$$

Hence

$$\gamma_1 = 52^\circ\, 57'\, 32'', \quad \gamma_2 = 102^\circ\, 22'\, 10''.35, \quad \gamma_3 = 119^\circ\;\; 7'\, 18'',$$
$$\gamma_4 = 122^\circ\;\; 5'\, 48'', \quad \gamma_5 = 58^\circ\, 25', \quad \gamma_6 = 57^\circ\, 21'\, 54'',$$
$$\gamma_7 = 77^\circ\, 48'\, 22''.7, \quad \gamma_8 = 123^\circ\, 49'\, 44''.2.$$

$F_x = +24.393$ lbs., $F_y = +290.29$ lbs., $F_z = +221.295$ lbs., $F_r = 365.84$ lbs.

$$\cos a = \frac{24.393}{365.84}, \quad \text{or} \quad a = 86^\circ\, 10'\, 36''; \quad \cos b = \frac{290.29}{365.84}, \quad \text{or} \quad b = 37^\circ\, 29'\, 14'';$$

$$\cos c = \frac{221.295}{365.84}, \quad \text{or} \quad c = 52^\circ\, 46'\, 48''.$$

∴ a = b

CHAPTER II.

STATICS—PARALLEL FORCES.

NON-CONCURRING FORCES. MOMENT OF A FORCE. LINE REPRESENTATIVE OF MOMENT OF A FORCE. RESOLUTION AND COMPOSITION OF MOMENTS. TWO NON-CONCURRING CO-PLANAR FORCES. TWO PARALLEL FORCES. MOMENT OF A COUPLE. LINE REPRESENTATIVE OF A COUPLE. COMPOSITION AND RESOLUTION OF COUPLES. CENTRE OF PARALLEL FORCES. PROPERTIES OF CENTRE OF MASS. CONDITIONS OF EQUILIBRIUM FOR PARALLEL FORCES.

Non-concurring Forces.*—In the preceding Chapter we have considered concurring forces, that is, forces which act at a common point. We shall now consider non-concurring parallel forces, that is, parallel forces which act at different points of a rigid body.

Moment of a Force.—Since force is proportional to the acceleration it causes, the *moment* of a force relative to any point or axis is defined precisely like moment of acceleration (page 60, Vol. I, *Kinematics*).

Hence the product of the magnitude of a force by the magnitude of the perpendicular let fall from any given point upon the direction of the force gives the magnitude of the *moment* of the force relative to that point.

The point is called the centre of moments. The perpendicular is called the *lever-arm* of the force.

The unit of moment of a force is then one poundal-foot, or one poundal with a lever-arm of one foot, or in gravitation units one pound-foot, or the weight of one pound with a lever-arm of one foot.

The same conventions as to sign are adopted as for moment of acceleration (page 60, Vol. I, *Kinematics*). Thus rotation counter-clockwise is positive (+) and clockwise negative (−).

The same principles must evidently hold for the moment of a force as for the moment of its acceleration. Hence

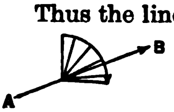
A force may be considered as acting at any point in its line of direction.

The algebraic sum of the moments of any number of forces is equal to the moment of their resultant (page 62, Vol. I, *Kinematics*).

Line Representative of Moment of a Force.—Since the moment of a force has thus magnitude and direction, it is a vector quantity and can be represented by a straight line like moment of acceleration.

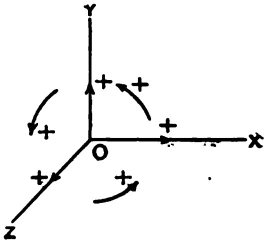
---

* The student should constantly refer in this portion of the work to the references in the text to *Kinematics of a Rigid System* (page 169, Vol. I), and if he has omitted that portion of the work should now take it in connection with *Statics*.



Thus the line AB represents by its length the magnitude of the moment. The plane of rotation is at right angles to this line. The direction of rotation is *clockwise* in this plane when we look *in the direction of the arrow*. When we speak of direction of a moment we mean the direction of its line representative.

Resolution and Composition of Moments.—The principles of pages 35, 36, Vol. I, *Kinematics*, hold good then for force moments as well as for acceleration moments (page 62, Vol. I, *Kinematics*), and we have the triangle and polygon of moments.



The signs of the line representatives of the components along the axes of X, Y, Z of a force moment follow the same rule as for components of acceleration (page 62, Vol. I, *Kinematics*). Hence components in the direction OX, OY, OZ are positive (+), in the opposite directions negative (—). If then we look along the line representatives of the components towards the origin O, the rotation is always counter-clockwise. Therefore rotation from X towards Y, Y towards Z, Z towards X is positive, in the opposite directions negative.

For polar co-ordinates directions away from the pole are positive, towards the pole negative.

Evidently, then, we measure angles in the plane XY, around from OX towards OY; in the plane YZ, around from OY towards OZ; in the plane ZX, around from OZ towards OX, as shown by the arrows in the figure.

Resultant of Two Non-concurring Co-planar Forces.*—Let the two forces F_1, F_2 act in the same plane at the points A, B of a rigid body, Fig. 1, in different directions, and let OF_r be the direction of the resultant F_r.

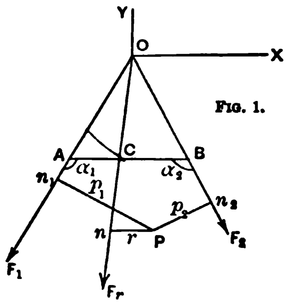


FIG. 1.

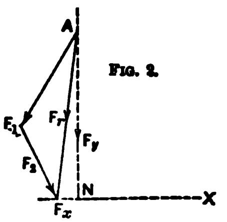


FIG. 2.

Take a point P anywhere in the plane of the forces and draw the lever-arms $Pn_1 = p_1$, $Pn_2 = p_2$, $Pn = r$.

Then, since the moment of the resultant with reference to any point is equal to the algebraic sum of the moments of the components, we have in general

$$F_r r = F_1 p_1 + F_2 p_2 . \quad \dots \dots \dots \quad (1)$$

[Regard must be paid to the signs. Thus if the forces are as represented in the figure, we have $+ F_1 p_1 - F_2 p_2$.]

* Compare page 179, for concurring angular accelerations, *Kinematics of a Rigid System*.

Since this holds good wherever we take the point P in the plane, let us suppose the point P at the intersection O of the given forces. For this point, the lever-arms p_1 and p_2 will be zero, the moments F_1p_1 and F_2p_2 will be zero, and hence $F_r r$ must be zero, or the lever-arm r is zero. We can therefore take the point O as the common point of application of F_1 and F_2 and the system reduces to two forces acting at the point O or to a system of concurring forces. Hence—

(1) *A force acting at any point of a rigid body can be considered as acting at any point in its line of direction.*

(2) *The resultant of two non-concurring co-planar forces lies in the plane of the forces and passes through the point of intersection of the forces.*

Position of the Resultant.—Draw the line AB intersecting the resultant F_r at the point C.

Let α_1 be the angle of F_1 with AB, and α_2 the angle of F_2 with AB. If we take moments about the point C, we have for the lever-arm of F_1, $AC \sin \alpha_1$, and for the lever-arm of F_2, $BC \sin \alpha_2$. From equation (1),

$$F_1 . AC \sin \alpha_1 = F_2 . BC \sin \alpha_2.$$

But $AC + BC = AB$. Hence

$$AC = \frac{F_2 . AB \sin \alpha_2}{F_1 \sin \alpha_1 + F_2 \sin \alpha_2}, \qquad BC = \frac{F_1 . AB \sin \alpha_1}{F_1 \sin \alpha_1 + F_2 \sin \alpha_2}. \quad (2)$$

We thus know the *position* of the resultant in the plane of the forces. (Compare page 179, *Kinematics of a Rigid System.*)

Magnitude and Direction of the Resultant.—The magnitude and direction of the resultant can now be found, precisely as for concurring forces.

Thus if we lay off F_1 and F_2 in the force polygon Fig. 2, AF_2 gives the magnitude and direction of the resultant F_r.

Take the rectangular axes OX and OY in the plane of the forces and let OX be parallel to AB. Let F_1 make the angle α_1 with OX, and β_1 with OY, and F_2 make the angle α_2 with OX, and β_2 with OY. Denote the algebraic sum of the components parallel to OX by F_x and parallel to OY by F_y. Then the equations of page 61 hold, and we have

$$\left.\begin{aligned} F_x &= F_1 \cos \alpha_1 + F_2 \cos \alpha_2; \\ F_y &= F_1 \cos \beta_1 + F_2 \cos \beta_2. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (3)$$

[Regard must be paid to the signs. Thus in the figure $F_2 \cos \alpha_2$ is positive, all the other terms are negative.]

If the resultant F_r makes the angles a and b with the axes of x and y, we have

$$\cos a = \frac{F_x}{F_r}, \qquad \cos b = \frac{F_y}{F_r}. \quad . \; . \; . \; . \; . \; . \quad (4)$$

Squaring and adding,

$$F_r = \sqrt{F_x^2 + F_y^2}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

In taking the summation indicated by (3), components in the direction OX or OY are positive, in the directions XO or YO negative.

If θ_1 is the angle of F_1 with the resultant, and θ_2 the angle of F_2

with the resultant, and θ the angle between F_1 and F_2, we have directly from the force polygon, Fig. 2,

$$\sin\theta_1 = \frac{F_2}{F_r}\sin\theta, \qquad \sin\theta_2 = \frac{F_1}{F_r}\sin\theta, \quad . \quad . \quad . \quad . \quad (6)$$

and

$$F_r = \sqrt{F_1^2 + F_2^2 \pm 2F_1F_2\cos\theta}, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where the (+) sign is used when θ is less then 90°, and the (—) sign when θ is greater than 90°.

The tangent of the angle a which the resultant makes with AB or OX is

$$\tan a = \frac{F_y}{F_x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From (6) and (7) we can find the magnitude and direction of the resultant directly if θ is known. If α_1 and α_2 are given, (3) and (5) give F_r, and (4) or (8) the direction.

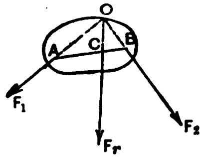
From (1) we have also

$$r = \frac{F_1p_1 + F_2p_2}{F_r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where regard must be had for the signs of F_1p_1 and F_2p_2 in any case.

From (9) for any given point P, for which p_1 and p_2 are known, we can locate the resultant by describing a circle with centre P and radius r, and drawing F_r tangent to this circle in the direction given by (6). (Compare page 180, *Kinematics of a Rigid System.*)

EXAMPLE.—*Two forces $F_1 = 20$ lbs. and $F_2 = 30$ lbs. act at points A, B of a rigid body, in the directions shown in the figure. The distance $AB = 2$ ft. and the angles $F_1AB = 120°$, $F_2BA = 150°$. Find the point of application C of the resultant, and its magnitude and direction.*



Ans. Cos $\alpha_1 = \sin\beta_1 = 0.5$, $\cos\alpha_2 = \sin\beta_2 = 0.866$, $\theta = 90°$. Hence

$$AC = \frac{30 \times 2 \times 0.5}{20 \times 0.866 + 30 \times 0.5} = 0.928 \text{ ft.}$$

$$\left.\begin{aligned} F_x &= -20 \times 0.5 + 30 \times 0.866 = +15.98; \\ F_y &= -20 \times 0.866 - 30 \times 0.5 = -32.32. \end{aligned}\right\} \tan a = -\frac{32.32}{15.98} = -2.022.$$

Or $BCF_r = 63° \ 41'$.

$$F_r = \sqrt{(15.98)^2 + (32.32)^2} = 36.05 \text{ lbs.}$$

We obtain the same result from equation (7) directly. Thus

$$F_r = \sqrt{20^2 + 30^2} = 36.05.$$

We also obtain from equation (6)

$$\sin\theta_1 = \frac{30}{36.05} = 0.832, \quad \text{or} \quad \theta_1 = 56° \ 19'.$$

Therefore $OCA = 180 - (60° + 56° \ 19') = 63° \ 41'$, as before.

Resultant of Two Parallel Forces.—This is but a special case of the preceding Article. Thus if two non-concurring forces are parallel, their intersection is at an infinite distance and α_1 and α_2

become equal, and $\theta = 0$. We have from equations (5) or (7), page 70,

$$F_r = F_1 + F_2,$$

where the forces F_1 and F_2 are to be taken with proper signs (+) in one direction and (−) in the opposite. From equation (2), page 69, we have

$$AC = \frac{F_2}{F_r} . AB, \quad BC = \frac{F_1}{F_r} . AB. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Multiplying the first by F_1 and the second by F_2, we have

$$F_1 . AC = F_2 . BC, \quad \text{or} \quad \frac{F_1}{F_2} = \frac{BC}{AC} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

To prove this independently, take C as centre of moments.

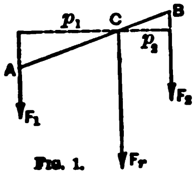


FIG. 1.

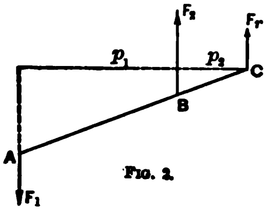


FIG. 2.

Then, whether the forces act in the same or in opposite directions, we have

$$F_1p_1 - F_2p_2 = 0, \quad \text{or} \quad F_1p_1 = F_2p_2,$$

where p_1 and p_2 are the lever-arms. But from similar triangles $\frac{p_1}{p_2} = \frac{AC}{BC}$. Hence

$$\frac{F_1}{F_2} = \frac{BC}{AC}.$$

We see from (1) that the distances AC and BC depend only upon the magnitudes of F_1 and F_2 and the distance AB between their points of application, and not at all upon the common direction of F_1 and F_2. Therefore if the forces F_1, F_2 are turned about A and B preserving their parallelism, or if the body is turned, the forces F_1 and F_2 having always the same direction and the same points of application, the resultant F_r will always pass through C. The point C is then *the point of application of the resultant.*

Hence, *the resultant of two parallel forces acting at the extremities of a rigid straight line is in their plane and equal in magnitude to their algebraic sum. It acts parallel to the forces in the direction of the greater force, and its point of application is on the straight line or the straight line produced, and divides it into segments inversely as the forces. Or the products of the forces into the adjacent segments are equal.* (Compare page 181, *Kinematics of a Rigid System.*)

This principle is known as the "law of the lever."

If we take the centre of moments at B and at A, we obtain directly equations (1).

COR. 1. When the forces act in the same direction, the resultant lies within the components. When the forces act in opposite

directions, the resultant lies without the components and on the side of the larger.

COR. 2. When the forces are equal and opposite, $F_r = 0$. Also, from (1), $AC = \infty$, $BC = \infty$, or the resultant is zero and acts at an infinite distance. That is, two equal and opposite parallel forces cannot have a single force as a resultant.

Such a system is called a **force couple.** (Compare page 182, *Kinematics of a Rigid System.*)

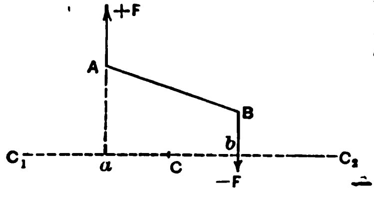
Since the resultant is zero, there is no force of translation, and the effect on AB is to cause rotation only. All tendency to rotation can be referred to forces forming such couples.

Moment of a Couple.*—From the last corollary, we see that a couple consists of two equal and parallel forces acting in opposite directions at different points of a rigid body.

The perpendicular distance between the directions of the forces is called the *arm* of the couple.

The product of the arm by one of the forces is the *moment* of the couple. This moment represents tendency to rotation of the rigid body.

Let the two equal, parallel and opposite forces, $+F$, $-F$, act at the points A and B of a rigid body. Draw any line C_1abC_2 at right angles to the direction of the forces.



Take any point C_1 on the left as a centre of moments. Then we have for the resultant moment about C_1, $F \,.\, C_1a - F(C_1a + ab) = -F \,.\, ab$.

For any point C_2 on the right, we have

$$F \,.\, C_2b - F(C_2b + ab = -F \,.\, ab.$$

For any point C between the forces,

$$-F \,.\, Ca - F \,.\, Cb = -F \,.\, ab.$$

The minus sign denotes clockwise rotation.

In general, *the moment of a couple about any point in its plane is constant and equal to the product of the arm by one of the forces.* (Compare page 186, *Kinematics of a Rigid System.*)

COR. 1. A couple may be turned round in any manner in its own plane without altering its effect, the arm ab being unchanged.

COR. 2. A couple may be removed to any position in its own plane without altering its effect, the arm ab being unchanged.

COR. 3. A couple may be transferred to any other plane parallel to its own plane without altering its effect.

COR. 4. All couples whose planes are parallel and moments equal, are equivalent.

COR. 5. Any couple may be replaced by another which shall be equivalent and have an arm of any given length.

COR. 6. We have for any point C_1 the resultant moment

$$F \,.\, C_1a - F(C_1a + ab).$$

If $C_1a = \infty$, then, since ab is insignificant with respect to C_1a, we have $F\infty - F\infty = 0$. The algebraic sum of the forces or the resultant force is also zero. The moment of a force is the algebraic sum of the moments of its components (page 67). The resultant there-

* Compare page 186, *Kinematics of a Rigid System.*

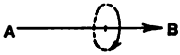
fore acts through any point where the moment sum of the components is zero. The resultant of a couple is therefore zero at an infinite distance in any direction in the plane of the couple. This is Cor. 2, page 72.

COR. 7. A couple cannot be replaced by a single force, but only by another equivalent couple.

COR. 8. A couple cannot be held in equilibrium by a single force, but only by another equivalent couple.

Line Representative of a Couple.—A line perpendicular to the plane of a couple is called the *axis* of the couple.

A couple can then be completely represented by a straight line. The length of the line represents the moment of the couple. The plane of the couple is at right angles to its line representative. The direction of rotation may be indicated by an arrow, so that looking along the line representative *in the direction of the arrow*, rotation is seen to be *clockwise*. Thus the line AB represents the magnitude of a couple causing rotation as indicated in a plane at right angles to the axis AB. The line representative coincides with the axis of rotation.



A couple is thus a vector quantity, like displacement, velocity, acceleration, moment, force, and the same principles apply as to composition and resolution of forces.

When we speak of the "direction of a couple" we mean the direction of its line representative.

Composition and Resolution of Couples.—We have then the "parallelogram and polygon of couples."

When couples are in the same plane, or parallel planes, their line representatives are all parallel. Hence the resultant of any number of couples in the same or in parallel planes equals the algebraic sum of the component couples.

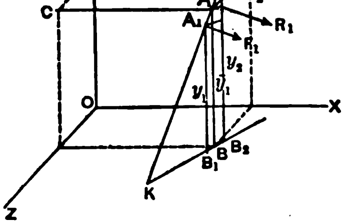
The resultant of two couples in different planes is given by the diagonal of the parallelogram constructed on the line representatives of the components, taken the other way round.

The resultant of any number of couples in different planes, the axes being all in the same plane, is given by the line which closes the polygon formed by the line representatives taken the other way round.

The line representatives can then be combined and resolved just like forces in general.

The action of a couple acting upon a rigid body is to cause angular acceleration of the body about an axis perpendicular to its plane.

Centre of Parallel Forces.*—Let F_1, F_2, F_3, etc., be any number of parallel forces acting at the points A_1, A_2, A_3, etc., of a rigid body.



Then the resultant F_r must be parallel to the forces and equal in magnitude to their algebraic sum, or

$$F_r = F_1 + F_2 + F_3 + \ldots = \Sigma F.$$

In taking the summation, all forces in one direction are (+), in the opposite direction (−).

Take any two of the parallel forces, as F_1, F_2, and draw a line

* Compare page 192, *Kinematics of a Rigid System*.

A_1A_2 through their points of application and produce it to intersection K with the plane of ZX. Drop perpendiculars A_1B_1, A_2B_2 to this plane and draw the line KB_1B_2 in this plane.

Now, from page 71, the resultant of F_1 and F_2 is $R_1 = F_1 + F_2$ and its point of application is at A on the line A_1A_2, such that

$$\frac{F_1}{F_2} = \frac{A_2A}{A_1A}.$$

Drop the perpendicular AB to the plane ZX. Then we have by similar triangles

$$\frac{A_2A}{A_1A} = \frac{B_2B}{BB_1}.$$

Denote the distances A_1B_1, A_2B_2 by y_1, y_2 respectively, and the distance AB, or the ordinate of the point of application of the resultant R_1 of F_1 and F_2, by $\bar{y}_1$. Then we have by similar triangles

$$\frac{B_2B}{BB_1} = \frac{y_2 - \bar{y}_1}{\bar{y}_1 - y_1}.$$

Hence

$$\frac{F_1}{F_2} = \frac{y_2 - \bar{y}_1}{\bar{y}_1 - y_1}, \quad \text{or} \quad \bar{y}_1 = \frac{F_1y_1 + F_2y_2}{F_1 + F_2}.$$

In the same way for three forces F_1, F_2, F_3 we can combine the resultant R_1 of F_1 and F_2 acting at the point A, with F_3. We thus obtain for the ordinate of the point of application of the resultant of three forces

$$\bar{y}_2 = \frac{F_1y_1 + F_2y_2 + F_3y_3}{F_1 + F_2 + F_3}.$$

In general, then, for any number of parallel forces we have for the ordinate $\bar{y}$ of the point of application of the resultant

$$\bar{y} = \frac{\Sigma Fy}{\Sigma F}. \quad \ldots \ldots \ldots \quad (1)$$

In precisely similar manner, if we denote the distances AC and AD of the point of application of the resultant from the planes of YZ and XY by $\bar{x}$ and $\bar{z}$, we have

$$\bar{x} = \frac{\Sigma Fx}{\Sigma F}; \quad \ldots \ldots \ldots \quad (2)$$

$$\bar{z} = \frac{\Sigma Fz}{\Sigma F}. \quad \ldots \ldots \ldots \quad (3)$$

Equations (1), (2) and (3) give the co-ordinates of the point of application of the resultant for any number of parallel forces. This point is called the *centre of parallel forces*.

We see that its position depends only upon the magnitude of the forces and the position of their points of application, and is independent of the common direction of the forces.

COR. 1. If $\bar{z}$ is zero, then z_1, z_2, etc., must be zero, and the parallel forces are co-planar and all lie in the plane XY. The centre is then given by (1) and (2). If $\bar{z}$ and $\bar{y}$ are zero, the points of application are all in the axis of X, and the centre is given by (2). (Compare page 192, *Kinematics of a Rigid System*.) If $\bar{x}$, $\bar{y}$ and $\bar{z}$ are

zero, the centre is at the origin. If $\bar{x}$ and $\bar{z}$ are zero, the centre is in the axis of Y and the points of application are all in the axis of Y, etc.

COR. 2. If a force equal and opposite to the resultant is applied at the centre of parallel forces, we have a system of parallel forces in equilibrium.

COR. 3. If a body has a motion of translation only, all the points of the body move in parallel paths with the same acceleration, if any, in the same direction at any instant. Let f be this common acceleration. Then if we consider the body to be composed of an indefinitely large number of indefinitely small particles of mass m_1, m_2, m_3, etc., the parallel forces on each of them are $F_1 = m_1f$, $F_2 = m_2f$, $F_3 = m_3f$, etc. The total resultant force in the common direction is then

$$R = m_1f + m_2f + m_3f + \text{etc.} = f(m_1 + m_2 + m_3 + \text{etc.});$$

or if the total mass $M = m_1 + m_2 + m_3 + \text{etc.}$,

$$R = fM.$$

Also, if the co-ordinates of the particles m_1, m_2, m_3, etc., are (x_1, y_1, z_1), (x_2, y_2, z_2), etc., and the co-ordinates of the point of application of the resultant are denoted by $\bar{x}$, $\bar{y}$, $\bar{z}$, we have, since the moment of the resultant is equal to the algebraic sum of the moments of the components,

$$R\bar{x} = fM\bar{x} = m_1fx_1 + m_2fx_2 + \text{etc.} = f\Sigma mx,$$

or

$$\bar{x} = \frac{\Sigma mx}{M}. \qquad (1)$$

In the same way we have

$$\bar{y} = \frac{\Sigma my}{M}, \qquad (2)$$

$$\bar{z} = \frac{\Sigma mz}{M}. \qquad (3)$$

The point given by equations (1), (2) and (3) coincides with the centre of mass of the body (page 17).

Hence, *the centre of mass of a body coincides with the point of application of the resultant of that system of parallel forces which acts upon all the particles of a translating body; that is, when each parallel particle force causes in the particle on which it acts the same acceleration in the same direction* (page 18).

Properties of the Centre of Mass.—We have then the following properties of the centre of mass:

1. The centre of mass coincides with the point of application of the resultant of that system of parallel forces which acts upon all the particles of a translating body.

2. Hence, inversely, if all the forces acting upon a rigid body reduce to a single resultant force acting at the centre of mass, the motion of the body is one of translation only.

3. The algebraic sum of the moments of the masses (page 19) of all the particles with reference to the centre of mass is zero (page 17).

If, then, the origin of co-ordinates is taken at the centre of mass, we have

$$\Sigma mx = 0, \quad \Sigma my = 0, \quad \Sigma mz = 0.$$

If polar co-ordinates are taken, and the pole is taken at the centre of mass, we have

$$\Sigma mr = 0,$$

where r is the distance of any particle from the centre of mass.

4. Since the attraction of the earth for a body at or above its surface, whose longest dimension is insignificant compared to the earth's radius, is practically an equal and parallel force on every equal particle of the body, the weight of the body in such case acts at its centre of mass, and a body acted upon only by its weight has a motion of translation only.

Hence the centre of mass is often erroneously called the "centre of gravity" (pages 18, 46).

5. In all positions of a rigid body about the centre of mass, the weight then passes practically through the centre of mass, because changing the direction of a system of parallel forces does not, as we have seen (page 74), change the point of application of the resultant, provided the points of application of the forces and their magnitudes are unchanged.

Hence if a rigid body free to move is supported at its centre of mass, it will be at rest in all positions about this centre, because in all positions we have two equal and opposite forces acting at the same point.

We can therefore locate the centre of mass of a rigid body by suspending it successively in two different positions. The two directions of the suspending string relative to the body must intersect practically at the centre of mass, since in each case, if the body is at rest, the centre of mass must be vertically under the point of suspension.

6. If a rigid body free to move is supported at a point vertically below the centre of mass, it will then be in equilibrium. But if the body be moved in any direction, however slightly, around the point of support, we shall have the weight of the body and the upward pressure on the support forming a couple causing the body to rotate away from its former position of equilibrium.

A body in such a position is said to be in *unstable equilibrium*.

If a rigid body is supported at any point vertically above the centre of mass, it will be in equilibrium also. If the body is moved in any direction however slightly around the point of support, we shall have a couple causing rotation towards the former position of equilibrium.

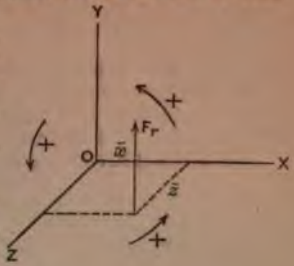
A body in such a position is said to be in *stable equilibrium*.

If the body is supported at the centre of mass, it will remain in equilibrium in any position about the point of support. It is then said to be in *indifferent equilibrium*.

7. The centre of mass may lie outside the limits of the body, as for example in the case of a circular ring or a spherical shell.

8. The motion of the centre of mass of a rigid body is the same as if the body were replaced by a particle of equal mass at the centre of mass, and all the forces acting upon the body were transferred to this particle without change in magnitude or direction (pages 18, 83).

Resultant Force and Couple for any Number of Parallel Forces.—Take the axis of Y parallel to the common direction of the parallel forces F_1, F_2, F_3, etc., and let these forces be applied at the points of a rigid body whose co-ordinates are (x_1, y_1, z_1), (x_2, y_2, z_2), etc.



Then the resultant will be the algebraic sum of all the forces, or

$$F_r = F_1 + F_2 + F_3 + \ldots = \Sigma F, \quad . \quad (1)$$

all forces acting in the direction OY being positive, and all in the opposite direction being negative in the algebraic sum.

The point of application $(\bar{x}, \bar{y}, \bar{z})$ of this resultant, or the centre of force, is given by

$$\bar{x} = \frac{\Sigma Fx}{\Sigma F}, \quad \bar{y} = \frac{\Sigma Fy}{\Sigma F}, \quad \bar{z} = \frac{\Sigma Fz}{\Sigma F}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Taking positive rotation in each co-ordinate plane as indicated in the figure from X to Y, Y to Z, Z to X, we have for the moment about the axis of X in the plane YZ

$$M_x = \bar{z}\Sigma F = \Sigma Fz, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and for the moment about the axis of Z in the plane XY

$$M_z = \bar{x}\Sigma F = \Sigma Fx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

There is no moment about the axis of Y, or $M_y = 0$. The line representatives of these moments are positive in the direction OX and OZ, negative in the opposite directions.

The resultant moment is then

$$M_r = \sqrt{M_x^2 + M_z^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The line representative of the resultant moment makes angles d, e and f with the axes of X, Y and Z whose cosines are given by

$$\cos d = \frac{M_x}{M_r}, \qquad \cos e = \frac{M_y}{M_r} = 0, \qquad \cos f = \frac{M_z}{M_r}. \quad . \quad . \quad . \quad (6)$$

Looking along this line representative *towards the origin*, the direction of rotation is always seen *counter-clockwise*.

Equilibrium of a Rigid Body.—If a rigid body acted upon by any number of forces applied at different points is in static equilibrium (page 58), all the forces must evidently reduce to two equal and opposite resultant forces acting in the same straight line. That is, *the algebraic sum of the moments of all the forces about every point in space must be zero.* Or, any one of the forces must be equal and opposite to the resultant of all the others and act in the same straight line with it. If any one of the forces is equal and opposite to the resultant of all the others, but does not act in the same straight line with it, we have molar equilibrium (page 58).

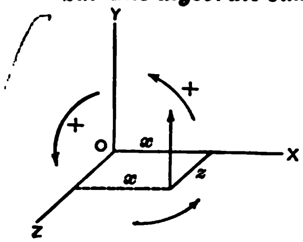
Conditions of Equilibrium of a Rigid Body acted upon by Parallel Forces.—If all the forces acting at different points of a rigid body are parallel, we have then for the necessary and sufficient conditions of static equilibrium:

1*st*. *The algebraic sum of the forces must be zero*, or

$$\Sigma F = 0. \qquad (1)$$

When this condition only is complied with, there is no resultant force, or any one of the forces is equal and opposite to the resultant of all the others, but does not necessarily act in the same straight line with it. (We have then molar equilibrium.) omit

2*d*. *The algebraic sum of the moments of the forces with reference to any two co-ordinate planes, parallel to the forces, must be zero.*



That is, if we take the common direction of the forces parallel to the axis of Y, and take the origin O as the centre of moments, we have the resultant moment $M_r = 0$, or

$$\Sigma Fx = 0, \quad \Sigma Fz = 0. \qquad (2)$$

When this condition only is complied with, there is no rotation about the origin O, or about any point in the axis OY.

The resultant then coincides with the axis OY. If this resultant is not also zero, there can be no static equilibrium. If it is zero, then the 1st condition is also fulfilled, and we have the algebraic sum of the moments of all the forces about every point in space, equal to zero.

In order, then, that there may be static equilibrium, both conditions (1) and (2) must be satisfied.

omit (COR. 1. If equilibrium, molar or static, exists for any one direction of the parallel forces, it will exist whatever the common direction, provided the magnitudes and points of application of the parallel forces are unchanged.)

COR. 2. If the parallel forces are co-planar, let their common plane be the plane of XY, and let their common direction be parallel to the axis of Y.

Then we have for the conditions of equilibrium

$$\Sigma F = 0; \qquad (1)$$

$$\Sigma Fx = 0. \qquad (2)$$

If the first condition alone is satisfied, we have molar equilibrium.

If the second alone is fulfilled, the resultant coincides with the axis of Y.

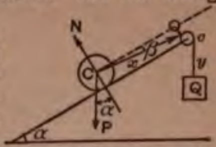
If both are fulfilled, we have the moment about every point in the plane zero, and hence static equilibrium.

EXAMPLES.

(1) *Show that the centre of mass of the perimeter of a triangle cannot coincide with the centre of mass of the triangular area, except in the case of an equilateral triangle.*

(2) *A mass P at rest on an inclined plane is attached to one end of a string which passes over a pulley at the top of the plane and supports at the other end a mass Q. The pressure of the plane upon P is normal to the plane. Show that when Q is moved vertically, the centre of mass of P and Q will neither rise nor fall.*

Ans. Let α be the angle of the plane with the horizontal. Let the string make the angle β with the plane.



The weight of P is the attraction of the earth for P. The tension of the string is the same as the weight of Q. Since P is at rest, the tension of the string Q, the weight P and the normal pressure N are in equilibrium and concur at the centre of mass C. Let l be the length of the string, and x the length of that portion of it, Cc, between the body and the pulley, and y that portion of it, cQ, between the pulley and the body Q. Then $x + y = l$, no matter where the body P is on the plane. The distance of the centre of mass of P and Q below the pulley is then

$$\frac{Px \sin(\alpha \pm \beta) + Qy}{P + Q},$$

where the (+) sign for β is taken when β is above and the (−) sign when β, as in the figure, is below the parallel to the plane through C.

But since P is at rest, the component of its weight parallel to Cc must be equal and opposite to the tension of the string Q. Hence $P \sin(\alpha \pm \beta) = Q$, and the distance of the centre of mass of P and Q below the pulley is $\frac{Q(x+y)}{P+Q} = \frac{Ql}{P+Q}$, which is *independent of the position of Q.*

(3) *Three masses of* 2, 3, 4 *ounces respectively lie in a straight line. The distance between the first and second is* 10 *inches, between the second and third* 5 *inches. Find the centre of mass.*

Ans. At the centre of mass of the middle mass.

(4) *Four masses of* 1, 2, 3, 4 *pounds are placed in order at equal distances one inch apart on a rod. Neglecting the rod, find the point at which they will balance.*

Ans. At the centre of mass of the third mass.

(5) *At the corners of a square, taken in order, are placed masses* 1, 3, 5, 7. *Find the centre of mass.*

Ans. If s is the length of a side of the square, the distance of the centre of mass from the side (1, 7) is $\frac{s}{2}$, and from the side (5, 7) $\frac{s}{4}$.

(6) *From a fixed horizontal rod are suspended a given number of equal masses by strings, the sum of the lengths of which is given. Find the distance of the centre of mass from the rod.*

Ans. If n is the number of masses and l the whole length of string used, the required distance is $\frac{l}{n}$.

(7) *Two masses support each other on two smooth inclined planes by means of a fine string passing over the common vertex of the planes. If the masses are moved, show that the centre of mass moves in a horizontal line.*

(8) *A solid right cone stands on a plane inclined at an angle of* 30° *to the horizon and is prevented from sliding. Find the height of the cone in terms of the radius of the base, in order that it may be on the point of overturning.*

Ans. $4r\sqrt{3}$.

(9) *A circular table weighing w lbs. has three equal legs at equidistant points on its circumference. The table is placed on a level floor. Neglecting the legs, find the smallest weight which, placed anywhere on the table, will just bring it to the point of overturning.*

Ans. w lbs.

Center of gravity of cone

(10) *If the table has four legs at equidistant points, find the least weight that will upset it.*

Ans. $2.4w$.

(11) *The centre of mass of a ladder weighing 50 lbs. is 12 ft. from one end, which is fixed. What force must a man apply at a distance of 6 ft. from this end to raise the ladder?*

Ans. 100 lbs.

(12) *Two parallel forces, acting in the same direction, are 17 and 33 lbs. respectively, and their points of application A, B are 8 ft. apart. Find the resultant and its intersection C with the line AB.*

Ans. $F_r = 50$ lbs. parallel to the forces

$$AC = 5.28 \text{ ft.}, \quad BC = 2.72 \text{ ft.}$$

(13) *Find the resultant and the point C when the forces in the preceding example act in opposite directions.*

Ans. $F_r = 16$ lbs. in the direction of the larger force

$$AC = 16.5 \text{ ft.}, \quad BC = 8.5 \text{ ft.}$$

(14) *Two parallel forces F_1, F_2 of 12.5 and 25 lbs. act in the same direction upon two points. The resultant acts at a distance of 4 ft. from F_1. What is the distance between the forces?*

Ans. 6 ft.

(15) *Resolve a force $F_r = 52$ lbs. into two parallel forces acting in the same direction, F_1 and F_2: (a) when the distances from F_r are 2 and 3 ft.; (b) when $F_1 = 20$ lbs. at a distance of 2 ft.*

Ans. (*a*) $F_1 = 31.2$ lbs., $F_2 = 20.8$ lbs.
(*b*) $F_2 = 32$ lbs. at a distance from F_r of 1.25 ft.

(16) *Resolve a force $F_r = 20$ lbs. into two parallel forces F_1, F_2, one of which, F_1, acts opposite to F_2: (a) when the forces are distant from F_r 8 and 3 ft.; (b) when F_1 is 30 lbs. and distant from F_r 6 ft.*

Ans. (*a*) $F_1 = 12$ lbs., $F_2 = 32$ lbs.
(*b*) $F_2 = 50$ lbs. at a distance of 3.6 ft.

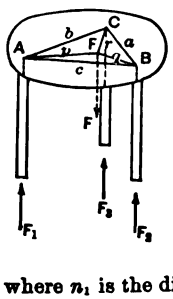
(17) *A beam of length l is supported at its ends. Parallel forces F_1, F_2, F_3 act upon it at right angles to its length, dividing the beam into the segments b, c, d and e. Find the pressures R_1 and R_2 at the supports at the left and right ends, neglecting the weight of the beam.*

Ans. $R_1 = \dfrac{F_1(l-b) + F_2(d+e) + F_3 e}{l}$, $R_2 = \dfrac{F_3(l-e) + F_2(b+c) + F_1 b}{l}$.

(18) *A table is supported by three legs at the points A, B, C. A load F is placed upon the table at the point F. Find the pressures on the legs.*

Ans. Let the upward pressures on the legs be F_1, F_2, F_3. Then

$$F_1 + F_2 + F_3 - F = 0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$



Let n_2 be the distance of F from the line AC, and h_2 the distance of B. Then, taking moments about AC,

$$Fn_2 - F_2 h_2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let n_3 be the distance of F from the line AB, and h_3 the distance of C. Then, taking moments about AB,

$$Fn_3 - F_3 h_3 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From these three equations we have

$$F_2 = \frac{Fn_2}{h_2}, \quad F_3 = \frac{Fn_3}{h_3}, \quad F_1 = \frac{Fn_1}{h_1},$$

where n_1 is the distance of F from BC, and h_1 the distance of A.

If the sides of the triangle ABC are a, b, c, and the angles BFC, CFA, AFB are α, β, γ, and the distances of F from A, B and C are p, q and r, we have

$$F_1 = \frac{Fn_1}{h_1} = F\frac{\frac{1}{2}n_1a}{\frac{1}{2}h_1a} = F\frac{qr \sin \alpha}{qr \sin \alpha + pr \sin \beta + pq \sin \gamma}.$$

In the same way we can find F_2 and F_3. If there are four legs, we have four unknown quantities and only three equations of condition. The problem is then indeterminate.

(19) *Find the resultant for a system of parallel co-planar forces given by*

$$\begin{array}{lll}
F_1 = +\ 83 \text{ lbs.}, & x_1 = +25 \text{ ft.}, & y_1 = +18 \text{ ft.}; \\
F_2 = +\ 20 \text{ ''} & x_2 = -10 \text{ ''} & y_2 = -15 \text{ ''} \\
F_3 = -\ 35 \text{ ''} & x_3 = +15 \text{ ''} & y_3 = -27 \text{ ''} \\
F_4 = -\ 72 \text{ ''} & x_4 = -31 \text{ ''} & y_4 = +17 \text{ ''} \\
F_5 = +120 \text{ ''} & x_5 = +23 \text{ ''} & y_5 = -19 \text{ ''}
\end{array}$$

Ans. $F_r = +66$ lbs., $\bar{x} = +77.15$ ft., $\bar{y} = -36.82$ ft.
If the forces are parallel to the axis of Y, $M_z = +5091.9$ lb.-ft.
If the forces are parallel to the axis of X, $M_z = +2430.12$ lb.-ft.
If we look along the line representative of the moment towards the origin, the rotation is seen counter-clockwise.

(20) *Find the resultant for the parallel-force system given by*

$$\begin{array}{llll}
F_1 = +\ 60 \text{ lbs.}, & x_1 = 0, & y_1 = 0, & z_1 = 0; \\
F_2 = +\ 70 \text{ ''} & x_2 = +1 \text{ ft.}, & y_2 = +2 \text{ ft.}, & z_2 = +3 \text{ ft}; \\
F_3 = -\ 90 \text{ ''} & x_3 = +2 \text{ ''} & y_3 = +3 \text{ ''} & z_3 = +4 \text{ ''} \\
F_4 = -150 \text{ ''} & x_4 = +3 \text{ ''} & y_4 = +4 \text{ ''} & z_4 = +5 \text{ ''} \\
F_5 = +200 \text{ ''} & x_5 = +4 \text{ ''} & y_5 = +5 \text{ ''} & z_5 = +6 \text{ ''}
\end{array}$$

Ans. $F_r = +90$ lbs., $\bar{x} = +2\frac{2}{3}$ ft., $\bar{y} = +3$ ft., $\bar{z} = +3\frac{1}{3}$ ft.
If the forces are parallel to the axis of Y, we have

$$M_x = +315 \text{ lb.-ft.}, \quad M_z = +240 \text{ lb.-ft.}, \quad M_r = 396 \text{ lb.-ft.}$$

The line representative making the angles with the axes of X, Y, Z given by

$$\cos d = +\frac{315}{396}, \qquad \cos e = 0, \qquad \cos f = +\frac{240}{396};$$

or

$$d = 322° \ 41' \ 41'', \qquad e = 90°, \qquad f = 52° \ 41' \ 41''.$$

If we look along the line representative towards the origin, the rotation is seen counter-clockwise.

CHAPTER III.

STATICS—NON-CONCURRING FORCES IN GENERAL.

COMPOSITION AND RESOLUTION OF FORCES AND COUPLES. CENTRAL AXIS OF A FORCE SYSTEM. CONDITIONS OF EQUILIBRIUM OF A RIGID BODY. ANALYTICAL DETERMINATION OF RESULTANT FORCE AND COUPLE FOR ANY NUMBER OF NON-CONCURRING FORCES IN SPACE. EQUIVALENT WRENCH. THE INVARIANT. COMPOSITION AND RESOLUTION OF WRENCHES.

In the preceding Chapter we have considered non-concurring forces when they are *parallel*. We shall now consider non-concurring forces in general, whatever their direction.

Composition and Resolution of Forces and Couples.—Let a force $AB = +F$ act at any point A of a rigid body.

If at any other point O of the body we introduce two equal and opposite forces, $Ob = +F$ and $Oa = -F$, each equal in magnitude to AB and parallel to it, the motion of the body is obviously unaffected by such introduction. We have then the force $AB = +F$ acting upon the body at A, reduced to an equal and parallel force $Ob = +F$, acting at any point O we please, and a couple consisting of AB and Oa.

The moment of this couple is the same for every point in its plane and equal to Fp, where p is the perpendicular distance between the forces AB and Ob (page 72). The action of this couple is to cause angular acceleration of the body about an axis perpendicular to its plane (page 72).

Since the motion of the point O is not affected by the introduction of the equal and opposite forces Ob and Oa, *the axis of rotation passes through O*. The motion of the body is therefore that of the point O at any instant, combined with rotation about the axis through O, perpendicular to the plane of the couple.

Hence (compare page 189, Vol. I, *Kinematics*), *A force F acting at any point of a rigid body can be resolved into an equal and parallel force at any point O of the body at a distance p from the line of direction of F, and a couple whose moment is Fp, whose plane is that of the forces, and whose axis of rotation passes through the point O perpendicular to this plane.*

Conversely, *The resultant of a force F acting at any point O of a rigid body and a couple whose moment is Fp and whose axis of rotation passes through the point O at right angles to the plane of the couple, is an equal and parallel force acting at a distance p in the plane of the couple.*

Cor. 1. Any number of forces acting at different points of a rigid body in different directions can then be reduced to a system

of concurring forces acting at any given point of the body, and a number of couples whose line representatives pass through that point. The forces can be reduced to a single resultant (page 58), and the couples can be reduced to a single resultant (page 73).

Hence any number of forces acting at different points of a rigid body in different directions can be reduced in general to a single force R acting at that point and a couple whose line representative passes through that point. The couple will vary with the point chosen. The force is the same no matter what point is chosen.

COR. 2. This resultant force R and couple whose moment is Rp can again be reduced to a single resultant equal and parallel force R at the distance p in the plane of the couple.

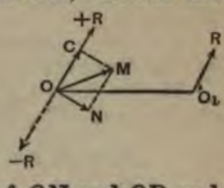
If this single resultant force R passes through the centre of mass, every point of the body has the same acceleration f in the same direction and the motion of the body is one of translation (page 75). The single resultant force is then $R = f\Sigma m$, or $f = \frac{R}{\Sigma m}$, where Σm is the mass of the body.

If this resultant force R does not pass through the centre of mass, it can be reduced to an equal and parallel force $R = f\Sigma m$ which does, and a couple whose plane is that of the forces and whose axis of rotation passes through the centre of mass. This couple then does not affect the acceleration of the centre of mass, which is therefore in both cases in the same direction and equal to $f = \frac{R}{\Sigma m}$.

Therefore, when a rigid body is acted upon by any number of forces applied at different points and acting in different directions, that is, *whatever the motion of the body may be, the motion of the centre of mass is precisely the same as if the body were replaced by a particle of equal mass at the centre of mass, and all the forces were transferred to this particle without change in direction or magnitude.*

Central Axis of a Force System.—Any number of forces acting at different points of a rigid body in different directions may be reduced to a single force and a couple whose axis is in the line of action of the force.

Let OR be the line representative of the force R, and OM the line representative of the couple M, passing through O, to which, as we have seen, any number of forces acting upon a rigid body may be reduced. Resolve OM into the components ON at right angles to OR, and OC along OR. The couple represented by ON can be replaced by the equal parallel and opposite forces $-R$ at O and $+R$ at a point O_1, the distance OO_1 being perpendicular to the plane of ON and OR and equal to $\frac{ON}{R}$. Then $-R$ and $+R$ at O balance, and the system is reduced to R at O_1 and the couple represented by OC, whose axis is parallel to R (compare page 191, Vol. I, *Kinematics of a Rigid System*). The couple represented by OC causes rotation of the body about the axis OC with a certain angular acceleration α, and therefore O_1 has the acceleration of translation $OO_1 . \alpha$.



But (page 190, Vol. I, *Kinematics of a Rigid System*) an angular acceleration α of a rigid system about any axis can be resolved into an equal angular acceleration about a parallel axis at any distance

OO_1 and an acceleration of translation $OO_1 . \alpha$ in a direction at right angles to the plane of the axis. The axis through O can then be shifted to O_1. The entire system of forces reduces then to the resultant force R at O_1 and a couple whose axis is in the line O_1R.

When this reduction is made, the line of action of the force is called the central axis of the **force system**, or **Pointsot's central axis**. (Compare page 191, Vol. I, *Kinematics of a Rigid System.*)

Sir R. S. Ball has given the name **wrench** to the resultant force and couple to which a given system of forces may be reduced when the line of action of the resultant force is the central axis.

COR. 1. Since OM is always greater than OC, it is evident that the magnitude of the resultant couple is less when its direction is that of the central axis than when it has any other direction.

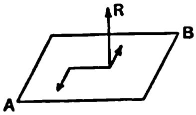
COR. 2. If ϕ is the angle between R and M, then denoting ON by N, and OC by C,

$$OO_1 = \frac{N}{R} = \frac{M \sin \phi}{R} C = M \cos \phi,$$

and this value of C gives the least value of the resultant moment. This is called *Pointsot's moment.*

Conditions of Equilibrium of a Rigid Body.—We have proved in the preceding Article that any forces acting on a rigid body can be reduced to a single resultant force R and a couple whose axis is parallel to that force or whose plane is at right angles to it.

In order, then, that *static* equilibrium may exist, R must be zero and the moment of the couple must be zero. Or, as we have stated (page 77), all the forces must evidently reduce to two equal and opposite forces acting in the same straight line. Hence, *the algebraic sum of the moments of all the forces about every point in space must be zero.* Any one of the forces, then, must be equal and opposite to the resultant of all the others and act in the same straight line with it. If any one of the forces is equal and opposite to the resultant of all the others, but does not act in the same line with it, we have *molar* equilibrium (page 58).



We have then two necessary and sufficient conditions for static equilibrium:*

1st. *The algebraic sum of the components of all the forces in each of any three rectangular directions must be zero.*

If the forces F_1, F_2, F_3, etc., make the angles $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, etc., with the co-ordinate axes, then we must have

$$\left.\begin{aligned} F_x &= F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + \text{etc.} = \Sigma F \cos \alpha = 0; \\ F_y &= F_1 \cos \beta_1 + F_2 \cos \beta_2 + \text{etc.} = \Sigma F \cos \beta = 0; \\ F_z &= F_1 \cos \gamma_1 + F_2 \cos \gamma_2 + \text{etc.} = \Sigma F \cos \gamma = 0. \end{aligned}\right\} \quad . \quad . \quad (1)$$

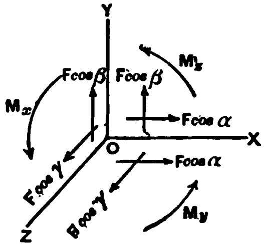
When these equations only are complied with, there is no resultant force and any one of the forces is equal and opposite to the resultant of all the others, but does not necessarily act in the same line with it. We have then molar equilibrium.

2d. *The algebraic sum of the component moments in each of any three given planes at right angles must be zero.*

---

* Compare page 199, Vol. I, *Kinematics of a Rigid System.*

If (x_1, y_1, z_1), (x_2, y_2, z_2), etc., are the co-ordinates of the points of application of the forces F_1, F_2, etc., then



$$\left.\begin{aligned} M_x &= \Sigma Fy\cos\gamma - \Sigma Fz\cos\beta = 0; \\ M_y &= \Sigma Fz\cos\alpha - \Sigma Fx\cos\gamma = 0; \\ M_z &= \Sigma Fx\cos\beta - \Sigma Fy\cos\alpha = 0. \end{aligned}\right\} \quad (2)$$

The figure shows the direction of positive rotation in each plane and of positive components $F\cos\alpha$, $F\cos\beta$, $F\cos\gamma$.

When these equations only are satisfied, there is no rotation about the origin O. The resultant then passes through O.

If this resultant is not also zero, there can be no static equilibrium. If it is zero, then the 1st condition is also satisfied and we have the algebraic sum of the moments of all the forces about every point in space equal to zero.

In order, then, that there may be static equilibrium, both conditions (1) and (2) must be fulfilled.

COR. 1. If the forces are all co-planar, let XY be their plane. Then $z = 0$, $\cos\gamma = 0$, and the general conditions of static equilibrium become

$$\left.\begin{aligned} F_x &= \Sigma F\cos\alpha = 0; \\ F_y &= \Sigma F\cos\beta = 0; \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

$$M_z = \Sigma Fx\cos\beta - \Sigma Fy\cos\alpha = 0. \; . \; . \; . \; . \; . \quad (2)$$

That is,

1st. *The algebraic sum of the components of the forces in each of any two rectangular directions in the plane of the forces must be zero.*

2d. *The algebraic sum of the moments of the forces about any point in this plane must be zero.*

If the first condition only is satisfied, we have molar equilibrium.

If the second only is satisfied, there is no rotation about the axis OZ. The resultant then coincides with this axis.

When this resultant is also zero, we have the algebraic sum of the moments of the forces about every point in the plane zero; both conditions are satisfied and there is static equilibrium.

COR. 2. If three non-concurring forces acting at different points of a rigid body are in equilibrium, their lines of direction produced must intersect in a common point and the forces must be co-planar.

For the resultant of any two must pass through their point of intersection and lie in their plane. The third force must be equal and opposite to this resultant and act in the same straight line.

COR. 3. If the forces are parallel, take their common direction parallel to the axis of Y. Then $\cos\alpha = 0$, $\cos\gamma = 0$, $\cos\beta = 1$, $F_x = 0$, $F_z = 0$, $F_y = \Sigma F$, and we have

$$\Sigma F = 0; \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

$$\Sigma Fx = 0, \quad \Sigma Fz = 0. \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

That is,

1st. *The algebraic sum of the forces must be zero.*

2d. *The algebraic sum of the moments of the forces with reference to any two co-ordinate planes parallel to the forces must be zero.*

These are the same conditions given on page 78.

If the first condition only is satisfied, we have molar equilibrium. If the second condition only is satisfied, the resultant passes through the origin and coincides with the axis of Y.

COR. 4. If the forces are parallel and co-planar, let their common plane be the plane of XY, and let them all be parallel to the axis of Y. Then we have

$$\Sigma F = 0; \quad \ldots \ldots \ldots \quad (1)$$

$$\Sigma Fx = 0. \quad \ldots \ldots \ldots \quad (2)$$

That is,

1st. *The algebraic sum of the forces must be zero.*

2d. *The algebraic sum of the moments of the forces about any point in their plane must be zero.*

Analytical Determination of Resultant Force and Couple for Any Number of Non-concurring Forces in Space.—(Compare page 197, Vol. I, *Kinematics of a Rigid System.*) Let any number of forces F_1, F_2, F_3, etc., acting at different points of a rigid body be given by (x_1, y_1, z_1), (x_2, y_2, z_2), etc., the origin being taken at some point of the rigid body. Let F_1 make with the co-ordinate axes of X, Y, Z the angles $(\alpha_1, \beta_1, \gamma_1)$ respectively; F_2, the angles $(\alpha_2, \beta_2, \gamma_2)$, etc. Then we have for the algebraic sum of the components parallel to the axes

$$\left.\begin{aligned} F_x &= F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + \ldots = \Sigma F \cos \alpha; \\ F_y &= F_1 \cos \beta_1 + F_2 \cos \beta_2 + \ldots = \Sigma F \cos \beta; \\ F_z &= F_1 \cos \gamma_1 + F_2 \cos \gamma_2 + \ldots = \Sigma F \cos \gamma. \end{aligned}\right\} \quad \ldots \quad (1)$$

Resultant Force.—If the resultant F_r makes the angles a, b, c with the axes, we have

$$F_r \cos a = F_x, \quad F_r \cos b = F_y, \quad F_r \cos c = F_z,$$

and hence the direction cosines are given by

$$\cos a = \frac{F_x}{F_r}, \quad \cos b = \frac{F_y}{F_r}, \quad \cos c = \frac{F_z}{F_r}. \quad \ldots \ldots \quad (2)$$

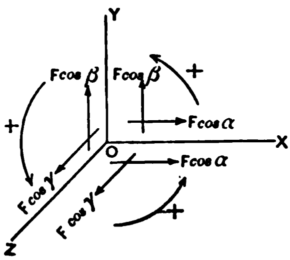
Squaring and adding, since $\cos^2 a + \cos^2 b + \cos^2 c = 1$,

$$F_r = \sqrt{F_x^2 + F_y^2 + F_z^2}. \quad \ldots \ldots \ldots \quad (3)$$

The magnitude and direction of the resultant force are thus determined.

There are precisely the same equations as for concurring forces, page 60.

Resultant Couple.—We can resolve each force, F_1, F_2, etc. (page 82), into an equal and parallel force acting at the origin O, and a couple causing a moment about O. Each couple can be resolved into component couples in the planes XY, YZ, ZX.



Taking, then, positive rotation as indicated by the figure in each plane, we have for the component moments in each plane about each axis (compare page 198, Vol. I, *Kinematics of a Rigid System*):

$$\left.\begin{array}{ll}
\left.\begin{array}{l}\text{about axis of } X\\ \text{in plane } YZ,\end{array}\right\} & M_x = \Sigma Fy \cos\gamma - \Sigma Fz \cos\beta;\\
\left.\begin{array}{l}\text{about axis of } Y\\ \text{in plane } ZX,\end{array}\right\} & M_y = \Sigma Fz \cos\alpha - \Sigma Fx \cos\gamma;\\
\left.\begin{array}{l}\text{about axis of } Z\\ \text{in plane } XY,\end{array}\right\} & M_z = \Sigma Fx \cos\beta - \Sigma Fy \cos\alpha.
\end{array}\right\} \quad . \quad . \quad (4)$$

The moment of the resultant couple is then given by

$$M_r = \sqrt{M_x^2 + M_y^2 + M_z^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and its direction cosines are given by

$$\cos d = \frac{M_x}{M_r}, \quad \cos e = \frac{M_y}{M_r}, \quad \cos f = \frac{M_z}{M_r}. \quad . \quad . \quad . \quad . \quad (6)$$

The axis passing through the origin is thus known in direction. The line representative coincides with this axis and is given in magnitude by (5). Looking along the line representative towards the origin, the direction of rotation is seen counter-clockwise.

The magnitude and direction of the resultant couple are thus known.

We have thus reduced the forces acting upon the body to a resultant force F_r acting at any point of the body taken as the origin O and a couple whose moment is M_r. The resultant force F_r is the same in magnitude and direction whatever point be taken. The moment M_r depends upon the point.

If r is the lever-arm of the resultant with reference to the origin O, we have

$$F_r r = M_r, \quad \text{or} \quad r = \frac{M_r}{F_r}.$$

Conditions of Equilibrium.—If the body is in static equilibrium, we must have

$$F_x = 0, \quad F_y = 0, \quad F_z = 0, \qquad \text{and also} \qquad M_x = 0, \quad M_y = 0, \quad M_z = 0.$$

We see from (3) that the first condition is fulfilled when $F_r = 0$, or the resultant force is zero. Therefore all the forces must reduce to two equal and opposite forces, or any one of the forces must be equal and opposite in direction to the resultant of all the others.

We see from (5) that the second condition is fulfilled when $M_r = 0$, that is, the two equal and opposite forces must act in the same line.

We have then for the equations of condition for equilibrium, from (1),

$$\left.\begin{array}{l}
\Sigma F \cos\alpha = 0;\\
\Sigma F \cos\beta = 0;\\
\Sigma F \cos\gamma = 0;
\end{array}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and from (4),

$$\left.\begin{array}{l}
\Sigma Fy \cos\gamma - \Sigma Fz \cos\beta = 0;\\
\Sigma Fz \cos\alpha - \Sigma Fx \cos\gamma = 0;\\
\Sigma Fx \cos\beta - \Sigma Fy \cos\alpha = 0.
\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

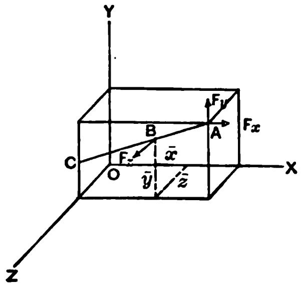
If equations (8) only are fulfilled, the two opposite resultant forces pass through the origin O, but unless (7) is also fulfilled they are not equal. (Compare page 199, Vol. I, *Kinematics of a Rigid*

System.) If (7) only is fulfilled, we have molar equilibrium (page 58). These are the same equations as on page 85.

Condition that there shall be a Single Resultant Force only.—If all the forces intersect at a single point, the moment at that point is zero, and all the forces acting upon the rigid body reduce then to a single resultant force at this point.

There is, however, one case in which the forces may not all intersect at a single point, and yet we may have a single resultant force. In this case all the forces must reduce to three, any two of which intersect, while the other, although it does not pass through their point of intersection, yet intersects their resultant.

Thus let the resultant forces parallel to the plane XY, F_x and F_y, intersect in a point A. We can then take them as acting at any point in the line of their resultant AC. Now suppose that the resultant force F_z parallel to the axis OZ intersects this resultant AC at B. Then we can take all three as acting at B, and thus have a single resultant force passing through B.



Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates of the point B. Then considering F_x, F_y, F_z acting at this point, we have

$$M_x = F_z\bar{y} - F_y\bar{z};$$
$$M_y = F_x\bar{z} - F_z\bar{x};$$
$$M_z = F_y\bar{x} - F_x\bar{y}.$$

If we multiply the first of these by F_x, the second by F_y, and the third by F_z and add, we have (compare page 200, Vol. I, *Kinematics of a Rigid System*)

$$F_xM_x + F_yM_y + F_zM_z = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Equation (9) gives the condition which must be satisfied in order that all the forces may reduce to a single resultant.

We have evidently for the projection of the line of this resultant on the co-ordinate planes

$$y = \frac{F_y}{F_x}x - \frac{M_z}{F_x}, \quad x = \frac{F_x}{F_z}z - \frac{M_y}{F_z}, \quad z = \frac{F_z}{F_y}y - \frac{M_x}{F_y}.$$

Co-planar Forces.—If the forces are all co-planar, take their plane as the plane of XY. Then $z = 0$, $\cos\gamma = 0$, and, from equations (1),

$$F_x = F_1\cos\alpha_1 + F_2\cos\alpha_2 + \ldots = \Sigma F\cos\alpha;$$
$$F_y = F_1\cos\beta_1 + F_2\cos\beta_2 + \ldots = \Sigma F\cos\beta;$$
$$F_z = 0;$$

and from equations (4),

$$M_x = 0, \quad M_y = 0, \quad M_z = \Sigma Fx\cos\beta - \Sigma Fy\cos\alpha.$$

We see, then, that equation (9) is satisfied. When the forces are co-planar, therefore, *they reduce to a single resultant.*

The equation of this resultant, if the plane of the forces is the plane of XY, is

$$y = \frac{F_y}{F_x}x - \frac{M_z}{F_x}.$$

The magnitude of the resultant is

$$F_r = \sqrt{F_x^2 + F_y^2}.$$

The resultant moment is M_z; and if r is the lever-arm of the resultant with reference to the origin,

$$r = \frac{M_z}{F_r}.$$

Parallel Forces.*—If the forces are all parallel, we have α, β, γ constant for all the forces. Hence from (1) and (2)

$$\left.\begin{aligned} F_x &= \cos\alpha\Sigma F = F_r \cos a; \\ F_y &= \cos\beta\Sigma F = F_r \cos b; \\ F_z &= \cos\gamma\Sigma F = F_r \cos c. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The resultant F_r must have the common direction of the parallel forces, or

$$a = \alpha, \quad b = \beta, \quad c = \gamma, \quad \text{and} \quad F_r = \Sigma F.$$

That is, the resultant F_r is equal to the algebraic sum of the forces and is parallel to them.

If we transfer the origin to any other point of the body whose co-ordinates are x', y', z', we have from (4), by putting $y - y'$, $x - x'$, $z - z'$ in place of y, x, z, and taking α, β, γ constant,

$$\left.\begin{aligned} M_x &= \cos\gamma\Sigma F(y-y') - \cos\beta\Sigma F(z-z') = \cos\gamma[\Sigma Fy - y'\Sigma F] - \cos\beta[\Sigma Fz - z'\Sigma F]; \\ M_y &= \cos\alpha\Sigma F(z-z') - \cos\gamma\Sigma F(x-x') = \cos\alpha[\Sigma Fz - z'\Sigma F] - \cos\gamma[\Sigma Fx - x'\Sigma F]; \\ M_z &= \cos\beta\Sigma F(x-x') - \cos\alpha\Sigma F(y-y') = \cos\beta[\Sigma Fx - x'\Sigma F] - \cos\alpha[\Sigma Fy - y'\Sigma F]. \end{aligned}\right\} . \quad (11)$$

If we substitute (11) and (10) in equation (9), we see that equation (9) is satisfied. All the forces reduce then to a single resultant force. The point of application of this force is given by the values of x', y', z' which make M_x, M_y, M_z zero. Hence the co-ordinates of the point of application of the resultant force are

$$\bar{x} = \frac{\Sigma Fx}{\Sigma F}, \quad \bar{y} = \frac{\Sigma Fy}{\Sigma F}, \quad \bar{z} = \frac{\Sigma Fz}{\Sigma F}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This point is the *centre of parallel forces* (page 73).

Equivalent Wrench.—(Compare page 201, Vol. I, *Kinematics of a Rigid System.*) We have seen (pages 83, 86) that all the forces acting upon a rigid body may be reduced to a resultant force F_r acting at any point of the body taken as the origin and a couple M_r causing rotation about an axis through that point. The resultant force F_r is the same in magnitude and direction no matter what point is taken. The couple M_r varies with the point. We have also seen (page 83) that this force and couple can be reduced to the resultant force F_r at a certain point and a resultant couple c_r whose axis is in the line of direction of F_r. The name *wrench* is given to this resultant force and couple; the axis is the *central axis;* the magnitude of the resultant force F_r is called the *intensity* of the wrench; the ratio of the moment c_r to the force F_r, or $\frac{c_r}{F_r}$, is evidently a linear magnitude and is called the *pitch*. It is the lever-arm of the couple which gives the moment c_r when the forces of the couple are equal to F_r.

A single force may thus be regarded as a wrench of zero pitch, a couple alone as a wrench of infinite pitch.

---

* Compare page 200, Vol. I, *Kinematics of a Rigid System.*

(1) The resultant force along the central axis is given by (3)

$$F_r = \sqrt{F_x^2 + F_y^2 + F_z^2}.$$

(2) The direction-cosines of the central axis are given by (2)

$$\cos a = \frac{F_x}{F_r}, \quad \cos b = \frac{F_y}{F_r}, \quad \cos c = \frac{F_z}{F_r}.$$

(3) The moment at every point resolved in a direction parallel to the central axis must be the same and equal to that in the direction of the central axis. Let c_r be the resultant moment along the central axis and let its components along the co-ordinate axes be c_x, c_y, c_z.

Take any point for which F_x, F_y, F_z and M_x, M_y, M_z are given as the origin, and let the co-ordinates of any point of the central axis be (x'', y'', z''). Then the components m_x, m_y, m_z of the moment at the origin due to the couple in the plane at right angles to the central axis are from equations (4), page 87,

$$\left.\begin{aligned} m_x &= F_z y'' - F_y z''; \\ m_y &= F_x z'' - F_z x''; \\ m_z &= F_y x'' - F_x y''. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (13)$$

We have then

$$M_x = c_x + m_x, \quad M_y = c_y + m_y, \quad M_z = c_z + m_z,$$

or

$$c_x = M_x - m_x, \quad c_y = M_y - m_y, \quad c_z = M_z - m_z. \quad . \ . \ . \quad (14)$$

Hence

$$c_r = (M_x - m_x) \cos a + (M_y - m_y) \cos b + (M_z - m_z) \cos c.$$

Inserting the values of the direction-cosines of the central axis, we obtain

$$c_r F_r = (M_x - m_x)F_x + (M_y - m_y)F_y + (M_z - m_z)F_z.$$

But since $m_x F_x + m_y F_y + m_z F_z = 0$, this becomes

$$c_r F_r = F_x M_x + F_y M_y + F_z M_z. \quad \ldots\ldots \quad (15)$$

We also have from (14)

$$c_r \cos a = c_x = M_x - m_x, \quad c_r \cos b = M_y - m_y, \quad c_r \cos c = M_z - m_z. \quad (16)$$

Hence from (13), inserting the values of the direction-cosines,

$$\frac{c_r}{F_r} = \frac{M_x + F_y z'' - F_z y''}{F_x} = \frac{M_y + F_z x'' - F_x z''}{F_y} = \frac{M_z + F_x y'' - F_y x''}{F_z}. \quad (17)$$

Equations (17) give the equation of the central axis.

From (15) we have

$$\frac{c_r}{F_r} = \frac{F_x M_x + F_y M_y + F_z M_z}{F_x^2 + F_y^2 + F_z^2}. \quad \ldots\ldots \quad (18)$$

This we have called the *pitch* (compare page 202, Vol. I, ***Kinematics of a Rigid System***). It is the lever-arm of the couple which gives the moment c_r when the forces of the couple are equal to F_r.

If we insert (17) in (16) and reduce, we have for the equation of the central axis

$$\left.\begin{aligned}\frac{1}{F_x}\left(x'' - \frac{F_yM_z - F_zM_y}{F_r^2}\right) &= \frac{1}{F_y}\left(y'' - \frac{F_zM_x - F_xM_z}{F_r^2}\right)\\ &= \frac{1}{F_z}\left(z'' - \frac{F_xM_y - F_yM_x}{F_r^2}\right)\end{aligned}\right\} \quad . \quad . \quad (19)$$

Therefore the central axis passes through a point whose coordinates are*

$$x'' = \frac{F_yM_z - F_zM_y}{F_r^2}, \quad y'' = \frac{F_zM_x - F_xM_z}{F_r^2}, \quad z'' = \frac{F_xM_y - F_yM_x}{F_r^2}. \quad (20)$$

If we substitute these values of x'', y'', z'' in (13) and (16), we have

$$\left.\begin{aligned}M_x &= c_r \cos a - F_r(z'' \cos b - y'' \cos c), & F_x &= F_r \cos a;\\ M_y &= c_r \cos b - F_r(x'' \cos c - z'' \cos a), & F_y &= F_r \cos b;\\ M_z &= c_r \cos c - F_r(y'' \cos a - x'' \cos b), & F_z &= F_r \cos a.\end{aligned}\right\} . \quad (21)$$

When, therefore, M_x, M_y, M_z, F_x, F_y, F_z are given for any point of the body, we can find the equivalent wrench, that is, the resultant force F_r, the direction of the central axis, and from (20) its position with reference to that point as an origin. We have also the couple c_r in the direction of the axis from (18).

On the other hand, if the position (x'', y'', z'') of the central axis is given, together with c_r and F_r, we can find M_x, M_y, M_z and F_x, F_y, F_z for the origin. The quantities F_x, F_y, F_z and M_x, M_y, M_z are called the *components* of the wrench. The wrench is known when these six quantities are known.

The Invariant.—(Compare page 203, Vol. I, *Kinematics of a Rigid System.*) From (15) we see that the quantity

$$F_xM_x + F_yM_y + F_zM_z$$

is always equal to F_rc_r, and is therefore invariable no matter what point is taken and whatever the values of F_x, F_y, F_z, that is, whatever the direction of the axes. This quantity is therefore called the **Invariant** of the components. Since F_r is also invariable whatever the direction of the axes, it may also be called the invariant of the couple.

If the invariant is zero, it follows that either F_r is zero or c_r is zero. The condition

$$F_xM_x + F_yM_y + F_zM_z = 0$$

is therefore the condition that there is no resultant force. or rotation only, or that there is no rotation and therefore a single resultant force only (see equation (9)).

Composition and Resolution of Wrenches.—(Compare page 203, Vol. I, *Kinematics of a Rigid System.*) If two wrenches are given, then by (21) we can find the six components of each wrench. Adding these two and two, we have the six components of the resultant wrench. Then by equations (2), (3), (15) and (20) the resultant wrench may be found.

* If the perpendicular from the origin to the central axis is p, then x'', y'', z'' are the projections of p upon the axes of X, Y, Z.

Conversely, we may resolve any given wrench into two wrenches in an infinite number of ways. Since a wrench is given by six components at any point, we have in the two wrenches twelve quantities at our disposal. Six of these are required to make the two wrenches equivalent to the given wrench. We may therefore in general satisfy six other conditions at pleasure.

Thus we may choose the axis of one wrench to be any given straight line we please.

Special Cases.—All cases are included by the general formulas (1) to (21) of the preceding Article.

(*a*) *For concurring forces in space*, take the origin as the point of concurrence. Then $M_x = 0$, $M_y = 0$, $M_z = 0$. If the concurring forces are in equilibrium, we have also $F_x = 0$, $F_y = 0$, $F_z = 0$.

(*b*) *For concurring co-planar forces*, take the origin as the point of concurrence. Then $M_x = 0$, $M_y = 0$, $M_z = 0$, and $F_z = 0$, $z = 0$.

(*c*) *For non-concurring co-planar forces*, take XY as the plane. Then $z = 0$, $F_z = 0$, $M_x = 0$, $M_y = 0$.

(*d*) *If one point of the body is fixed*, take that point as origin. Then since there can be no translation, $F_x = 0$, $F_y = 0$, $F_z = 0$.

(*e*) *If an axis parallel to X is fixed*, there can only be translation along this axis and rotation about it. Hence $F_y = 0$, $F_z = 0$, $M_y = 0$, $M_z = 0$.

(*f*) *If two points are fixed*, there can be no translation, but only rotation. If we take the axis of X through the points, we have $F_x = 0$, $F_y = 0$, $F_z = 0$, $M_y = 0$, $M_z = 0$.

(*g*) *If one point is always in the plane XY*, the body can have no translation parallel to z. Hence $F_z = 0$.

(*h*) *If three points not in the same straight line are confined to the plane XY*, we have rotation about Z only and no translation along Z. Hence $F_z = 0$, $M_x = 0$, $M_y = 0$.

(*i*) *If two axes parallel to X are fixed*, we can only have translation parallel to X. Hence $F_y = 0$, $F_z = 0$, and $M_x = 0$, $M_y = 0$, $M_z = 0$.

(*j*) *If the forces are all parallel to Y*, there is translation parallel to Y only, and rotation only about Z and X. Hence $F_x = 0$, $F_z = 0$, $F_y = 0$.

EXAMPLES.

(1) *Let a rigid body be acted upon by the co-planar forces*

$$F_1 = 50 \text{ lbs.},\quad F_2 = 30 \text{ lbs.},\quad F_3 = 70 \text{ lbs.},\quad F_4 = 90 \text{ lbs.},\quad F_5 = 120 \text{ lbs.}$$

acting at the points given by

$$x_1 = +5 \text{ ft.},\ y_1 = +10 \text{ ft.};\quad x_2 = +9 \text{ ft.},\ y_2 = +12 \text{ ft.};$$
$$x_3 = +17 \text{ ft.},\ y_3 = +14 \text{ ft.};\quad x_4 = +20 \text{ ft.},\ y_4 = +13 \text{ ft.};$$
$$x_5 = +15 \text{ ft.},\ y_5 = +8 \text{ ft.}$$

Let the forces make angles with the axes of X and Y, given by

$$\alpha_1 = 70^\circ,\ \beta_1 = 20^\circ;\quad \alpha_2 = 60^\circ,\ \beta_2 = 150^\circ;\quad \alpha_3 = 120^\circ,\ \beta_3 = 30^\circ;$$
$$\alpha_4 = 150^\circ,\ \beta_4 = 120^\circ;\quad \alpha_5 = 90^\circ,\ \beta_5 = 0^\circ.$$

Find the resultant, etc. (Compare Ex. (13), Vol. I, page 207.)

Ans. We have (page 86) for the components parallel to the axes of X and Y:

$$F_x = 50 \cos 70^\circ + 30 \cos 60^\circ - 70 \cos 60^\circ - 90 \cos 30^\circ = -80.842 \text{ lbs.};$$
$$F_y = 50 \cos 20^\circ - 30 \cos 30^\circ + 120 + 70 \cos 30^\circ - 90 \cos 60^\circ = +156.626 \text{ lbs.};$$
$$F_z = 0.$$

The resultant is given in magnitude by

$$F_r = \sqrt{F_x^2 + F_y^2} = 176.259 \text{ lbs.},$$

and its direction-cosines by

$$\cos a = \frac{F_x}{F_r} = \frac{-80.842}{176.259}, \quad \text{or} \quad a = 117° \, 18' \, 1'';$$

$$\cos b = \frac{F_y}{F_r} = \frac{+156.626}{176.259}, \quad \text{or} \quad b = 27° \, 18' \, 1''.$$

We have from equation (4), page 87,

$$\Sigma F_x \cos \beta = +50 \cos 20° \times 5 - 30 \cos 30° \times 9 + 70 \cos 30° \times 17 - 90 \cos 60° \times 20 + 120 \times 15 = +1981.670 \text{ lb.-ft.};$$

$$\Sigma F_y \cos \alpha = +50 \cos 70° \times 10 + 30 \cos 60° \times 12 - 70 \cos 60° \times 14 - 90 \cos 30° \times 18 = -1152.245 \text{ lb.-ft.}$$

$$M_x = 0, \; M_y = 0, \; M_z = \Sigma F_x \cos \beta - \Sigma F_y \cos \alpha = +3083.915 \text{ lb.-ft.}$$

Since, then, equation (9), page 88,

$$F_x M_x + F_y M_y + F_z M_z = 0,$$

is satisfied, the forces reduce to a single resultant force.

The moment of this resultant force relative to the origin is

$$M_r = \sqrt{M_x^2 + M_y^2 + M_z^2} = M_z = +3083.915 \text{ lb.-ft.}$$

Its lever-arm is

$$r = \frac{M_r}{F_r} = \frac{3083.915}{176.259} = 17.5 \text{ ft.}$$

The equation of the line of direction of the resultant (page 88) is

$$y = \frac{F_y}{F_x} x - \frac{M_z}{F_x} = -1.95x + 38.14.$$

The co-ordinates of the point of application of the resultant are given from equations (12), page 89:

$$\bar{x} = \frac{\Sigma F_x \cos \beta}{F_y} = \frac{+1981.67}{+156.626} = +12\tfrac{1}{4} \text{ ft.};$$

$$\bar{y} = \frac{\Sigma F_y \cos \alpha}{F_x} = \frac{-1152.245}{-80.842} = +14.25 \text{ ft.}$$

(2) *Find the resultant, etc., for the force system acting on a rigid body given by*

$F_1 =$ 50 lbs.;	$\alpha_1 = 60°$,	$\beta_1 = 40°$,	γ_1 acute;
$F_2 =$ 70 "	$\alpha_2 = 65°$,	$\beta_2 = 45°$,	γ_2 obtuse;
$F_3 =$ 90 "	$\alpha_3 = 70°$,	$\beta_3 = 50°$,	γ_3 acute;
$F_4 =$ 120 "	$\alpha_4 = 75°$,	$\beta_4 = 55°$,	γ_4 obtuse.
$x_1 = 0$,	$y_1 = 0$,	$z_1 = 0$;	
$x_2 = +1$ ft.,	$y_2 = +4$ ft.,	$z_2 = +7$ ft.;	
$x_3 = +2$ "	$y_3 = +5$ "	$z_3 = +8$ "	
$x_4 = +3$ "	$y_4 = +6$ "	$z_4 = +9$ "	

(Compare Ex. (15), Vol. I, page 208.)

Ans. We find the angles γ by the formula, Vol. I, page 12,

$$\cos^2 \gamma = -\cos(\alpha + \beta)\cos(\alpha - \beta).$$

Then from page 86 we have

$$F_x = +116.423 \text{ lbs.}, \quad F_y = +214.480 \text{ lbs.}, \quad F_z = -51.057 \text{ lbs.}$$

Therefore the resultant is

$$F_r = \sqrt{F_x^2 + F_y^2 + F_z^2} = +249.325 \text{ lbs.},$$

and its direction-cosines are given by

$$\cos a = \frac{F_x}{F_r}, \quad \cos b = \frac{F_y}{F_r}, \quad \cos c = \frac{F_z}{F_r},$$

or

$$a = 62^\circ\ 9'\ 48'', \quad b = 30^\circ\ 39'\ 20'', \quad c = 101^\circ\ 49'.$$

We also have for the moments from equation (4), page 87,

$$M_x = -1838.604, \quad M_y = +928.947, \quad M_z = -86.903 \text{ lb.-ft.}$$

The resultant moment about the origin is

$$M_r = \sqrt{M_x^2 + M_y^2 + M_z^2} = +2061.789 \text{ lb.-ft.},$$

and the direction-cosines of its line representative are given by

$$\cos d = \frac{M_x}{M_r}, \quad \cos e = \frac{M_y}{M_r}, \quad \cos f = \frac{M_z}{M_r},$$

or

$$d = 153^\circ\ 5'\ 40'', \quad e = 63^\circ\ 14'\ 15'', \quad f = 92^\circ\ 24'\ 56''.$$

Looking along this line representative towards the origin, the direction of rotation is seen counter-clockwise.

The equations of the projection of the resultant on the co-ordinate planes are

$$y = 1.885x + 0.746, \quad x = -2.28z + 18.19, \quad z = -0.238y - 8.57.$$

We see that

$$F_xM_x + F_yM_y + F_zM_z$$

does not in this case equal zero. Hence, page 88, the forces do not reduce to a single resultant force, but to a resultant force along the central axis and a couple whose axis is the central axis.

The resultant force along the central axis is, as already found, $F_r = 249.325$ lbs., and its angles with the co-ordinate axes are as already found.

The co-ordinates of the central axis are given by equation (20), page 91,

$$x'' = \frac{F_yM_z - F_zM_y}{F_r^2} = +0.463 \text{ ft.}, \quad y'' = \frac{F_zM_x - F_xM_z}{F_r^2} = +1.678 \text{ ft.},$$

$$z'' = \frac{F_xM_y - F_yM_x}{F_r^2} = +8.08 \text{ ft.}$$

The resultant couple c_r is given by equation (15), page 90,

$$c_r = \frac{F_xM_x + F_yM_y + F_zM_z}{F_r} = -41.624 \text{ lb.-ft.}$$

The direction cosines of its line representative are the same as for the resultant F_r, and looking along this line representative towards the origin the rotation is seen counter-clockwise.

The components of c_r are given by equation (16), page 90,

$$c_x = c_r \cos a = -19.481 \text{ lb.-ft.}, \quad c_y = c_r \cos b = -35.806 \text{ lb.-ft.},$$

$$c_z = c_r \cos c = +8.5288 \text{ lb.-ft.}$$

(3) *In the preceding example find what the co-ordinates x_4, y_4, z_4 of the force F_4 = 120 lbs. must be in order that all the forces may reduce to a single resultant.* (Compare Ex. 16, page 362.)

Ans. We evidently have F_x, F_y, F_z, F_r and the angles a, b, c unchanged, since changing the point of application of F_4 without changing its direction or magnitude has no effect on the magnitude of the resultant or its direction.

We have then

$$\left.\begin{aligned} M_x &= -659.571 - 93.262y_4 - 68.829z_4; \\ M_y &= +369.629 + 31.059z_4 + 93.262x_4; \\ M_z &= -107.036 + 68.829x_4 - 31.059y_4. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We have as the equation of condition for a single resultant, equation (9), page 88,

$$F_xM_x + F_yM_y + F_zM_z = 0,$$

or

$$116.423M_x + 214.48M_y - 51.057M_z = 0,$$

or

$$M_x + 1.842M_y - 0.4386M_z = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) we obtain

$$(M_x + 659.571)31.059 + (M_y - 369.629)68.829 = (M_z + 107.036)93.262,$$

or

$$M_x + 2.216M_y - 3.003M_z = +481.034. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (2) and (3) we obtain

$$0.374M_y - 2.564M_z = +481.034.$$

If we retain for M_y its value in the preceding example, $+928.947$ lb.-ft., we shall have

$$M_z = -\quad 52.108 \text{ lb.-ft.,}$$
$$M_x = -1733.95 \quad \text{"}$$

If we substitute these values in (1), we obtain

$$93.262y_4 + 68.829z_4 = +1074.4;$$
$$31.059z_4 + 93.262x_4 = +\ 559.308;$$
$$68.829x_4 - 31.059y_4 = +\quad 54.984.$$

Hence

$$x_4 = -0.333z_4 + \ 5.997;$$
$$y_4 = -0.738z_4 + 11.520.$$

If then we assume $z_4 = 0$, we have

$$x_4 = +5.997, \quad y_4 = +11.520.$$

(4) *Using the values of the preceding example, find the point of application of the resultant.* (Compare Ex. 17, Vol. I, page 210.)

Ans. We have

$$F_x = +116.423 \text{ lbs.,} \quad F_y = +214.480 \text{ lbs.,} \quad F_z = -51.057 \text{ lbs.,}$$
$$F_r = +249.325 \text{ lbs.;}$$

$$a = 62^\circ\ 9'\ 48'', \quad b = 30^\circ\ 39'\ 20'', \quad c = 101^\circ\ 49';$$

$$M_x = -1733.975 \text{ lb.-ft.,} \quad M_y = +928.947 \text{ lb.-ft.,} \quad M_z = -52.108 \text{ lb.-ft.,}$$
$$M_r = +1967.823 \text{ lb.-ft.;}$$

$$d = 151^\circ\ 47', \quad e = 61^\circ\ 49'\ 58'', \quad f = 91^\circ\ 31'\ 3''.$$

The co-ordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of the point of application of the resultant are given (page 89) by

$$-1733.975 = F_z\bar{y} - F_y\bar{z} = -\ \ 51.057\bar{y} - 214.480\bar{z};$$
$$+\ 928.947 = F_x\bar{z} - F_z\bar{x} = +116.423\bar{z} +\ 51.057\bar{x};$$
$$-\ \ 52.108 = F_y\bar{x} - F_x\bar{y} = \ \ 214.480\bar{x} - 116.423\bar{y}.$$

Hence we obtain

$$\bar{x} = -2.2802\bar{z} + 18.194,$$
$$\bar{y} = -4.2008\bar{z} + 33.961.$$

If we assume $\bar{z} = 0$, we have then

$$\bar{x} = +18.194 \text{ ft.},\quad \bar{y} = +33.961 \text{ ft.}$$

If we introduce, then, a fifth force, $F_5 = +249.325$ lbs., whose direction makes with the axes the angles

$$\alpha_5 = 117°\ 50'\ 12'',\quad \beta_5 = 149°\ 20'\ 40'',\quad \gamma_5 = 78°\ 11',$$

acting at a point whose co-ordinates are $\bar{x} = +18.194$ ft. and $\bar{y} = 33.961$ ft., $\bar{z} = 0$, we have a system of forces in equilibrium.

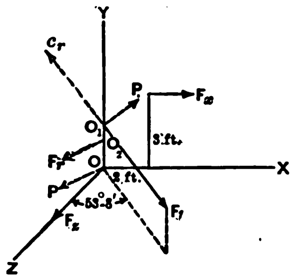
(5) *Find the resultant, etc., for the parallel-force system given by*

$F_1 = +\ 60$ lbs.;	$x_1 = 0$,	$y_1 = 0$,	$z_1 = 0$;
$F_2 = +\ 70$ "	$x_2 = +1$ ft.,	$y_2 = +2$ ft.,	$z_2 = +3$ ft.;
$F_3 = -\ 90$ "	$x_3 = +2$ "	$y_3 = +3$ "	$z_3 = +4$ "
$F_4 = -150$ "	$x_4 = +3$ "	$y_4 = +4$ "	$z_4 = +5$ "
$F_5 = +200$ "	$x_5 = +4$ "	$y_5 = +5$ "	$z_5 = +6$ "

Ans. $F_r = \Sigma F = +90$ lbs.;

$$\bar{x} = \frac{\Sigma Fx}{F_r} = +2\tfrac{2}{3} \text{ ft.},\quad \bar{y} = \frac{\Sigma Fy}{F_r} = +3 \text{ ft.},\quad \bar{z} = \frac{\Sigma Fz}{F_r} = +3\tfrac{1}{3} \text{ ft.}$$

tule (6) *A rigid body is acted upon by two forces $F_1 = 40$ lbs. and $F_2 = 30$ lbs. applied at points whose co-ordinates are $x_1 = 2$ ft., $y_1 = 3$ ft., $z_1 = 0$, and $x_2 = 0$, $y_2 = 0$, $z_2 = 0$, and making angles with the axes given by $\alpha_1 = 0°$, $\beta_1 = 90°$, $\gamma_1 = 90°$, and $\alpha_2 = 90°$, $\beta_2 = 90°$, $\gamma_2 = 0$. Find the equivalent wrench.*



Ans. (page 89). We have the components of the wrench

$$F_x = +40 \text{ lbs.},\quad F_y = 0,\quad F_z = +30 \text{ lbs.};$$
$$M_x = 0,\qquad M_y = 0,\quad M_z = -120 \text{ lb.-ft.}$$

The resultant force is $F_r = 50$ lbs., and its direction-cosines are

$$\cos a = \frac{+40}{50},\quad \cos b = 0,\quad \cos c = \frac{+30}{50},$$

or

$$a = 36°\ 52',\quad b = 90°,\quad c = 53°\ 8'.$$

The central axis coincides with F_r and makes the same angles with the axes. It passes through the point whose co-ordinates are

$$x'' = 0,\quad y'' = +1.92 \text{ ft.} = OO_1,\quad z'' = 0.$$

The moment of the couple whose axis coincides with the central axis is

$$c_r = -72 \text{ lb.-ft.}$$

The minus sign indicates that the line representative acts opposite to F_r, that is, its components in the direction of the axes are

$$c_x = -57.6 \text{ lb.-ft.}, \quad c_y = 0, \quad c_z = -120 \text{ lb.-ft.}$$

Its line representative acts then in the opposite direction from F_r and makes angles with the axes given by

$$a = 143° \, 8', \quad b = 90°, \quad c = 126° \, 2'.$$

Looking along this line representative towards the origin, rotation is seen counter-clockwise.

The moment c_r can be replaced by the two equal and opposite forces P, P acting at O_1 and O as shown in the figure, each equal to $\frac{c_r}{y''} = \frac{72}{1.92} = 37.5$ lbs.

If O is the centre of mass, then since the motion of the centre of mass is the same as if the entire mass of the body were concentrated at the centre of mass and all the forces acted at that point (page 83), the motion of O is the same as if F_r acted upon the entire mass M concentrated at O. The acceleration of O is then $\bar{f} = \frac{F_r}{M}$. The motion of the body is then a motion of translation due to F_r acting at the centre of mass and an angular acceleration α, due to the moment c_r, or the two equal opposite forces P, P acting at O_1 and O about an axis through O coinciding with the direction of F_r.

If we divide c_r by F_r, we obtain $\frac{c_r}{F_r} = \frac{72}{50} = 1.44$ ft. That is, we can replace the moment c_r by two equal and opposite forces F_r, F_r acting at O_1 and O_2. The distance O_1O_2 is then the *pitch*.

(7) *All the forces acting upon a rigid body reduce to a resultant force $F_r = 10$ lbs. acting at a given point and a couple whose moment is $M_r = 8$ lb.-ft. causing rotation about an axis through the point, which makes an angle of 45° with the direction of F_r. Find the equivalent wrench.*

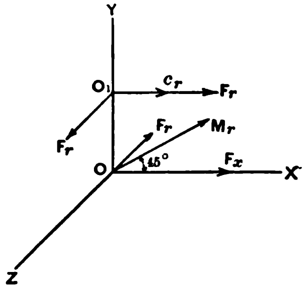
Ans. Take the direction of F_r as the axis of X, and the plane of F_r and the axis as the plane of XZ, and the point as origin. Then the components of the equivalent wrench are

$$F_x = +10 \text{ lbs.}, \quad F_y = 0, \quad F_z = 0;$$

$$M_x = +\frac{8}{\sqrt{2}} \text{ lb.-ft.},$$

$$M_y = 0,$$

$$M_z = -\frac{8}{\sqrt{2}} \text{ lb.-ft.}$$



We have then for the intensity of the wrench

$$F_r = 10 \text{ lbs.},$$

making the angles with the co-ordinate axes

$$a = 0, \quad b = 90°, \quad c = 90°.$$

The central axis passes through the point O_1 whose co-ordinates are

$$x'' = 0, \quad y'' = +\frac{8}{10\sqrt{2}} \text{ ft.} = OO_1, \quad z'' = 0,$$

and coincides with the direction of F_r.

The moment of the couple whose axis coincides with the central axis is

$$c_r = +\frac{8}{\sqrt{2}} \text{ lb.-ft.} = M_x.$$

The (+) sign indicates that the line representative acts in the same direction as F_r, that is, its components in the direction of the axes are

$$c_x = +\frac{8}{\sqrt{2}} \text{ lb.-ft.,} \quad c_y = 0, \quad c_z = 0.$$

Its line representative acts then in the same direction as F_r and makes the same angles with the axes as F_r. Looking along this line representative towards the origin, rotation is seen counter-clockwise.

The moment c_r can be replaced by two equal and opposite forces each equal to F_r acting at a distance given by

$$\frac{c_r}{F_r} = \frac{8}{10\sqrt{2}} \text{ ft.}$$

Since this distance is equal to $y'' = OO_1$, the *pitch* is in this case OO_1.

J. R. Brussel

CHAPTER IV.

STATICS—NON-CONCURRING CO-PLANAR FORCES.

CONDITIONS OF EQUILIBRIUM OF A RIGID BODY ACTED UPON BY NON-CONCURRING CO-PLANAR FORCES. DETERMINATION OF THE REACTIONS OF A FRAMED STRUCTURE. DETERMINATION OF THE STRESSES IN A FRAMED STRUCTURE. SUPERFLUOUS MEMBERS. CRITERION FOR SUPERFLUOUS MEMBERS.

Conditions of Equilibrium of a Rigid Body Acted Upon by Non-concurring Co-planar Forces.—We have seen (page 84) that when a rigid body is acted upon by any number of non-concurring co-planar forces, the conditions of static equilibrium are two, viz.:

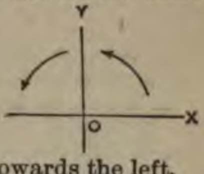
1st. *The algebraic sum of the components of the forces in each of any two rectangular directions in the plane of the forces must be zero.*

Hence if the forces F_1, F_2, etc., make the angles (α_1, β_1), (α_2, β_2), etc., with the co-ordinate axes, we must have

$$F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + F_3 \cos \alpha_3 + \ldots = \Sigma F \cos \alpha = 0; \quad . \quad (1)$$

$$F_1 \cos \beta_1 + F_2 \cos \beta_2 + F_3 \cos \beta_3 + \ldots = \Sigma F \cos \beta = 0. \quad . \quad (2)$$

When these equations are complied with there is no resultant force, and any one of the forces is equal and opposite to the resultant of all the others, but does not necessarily act in the same straight line with it. We have then molar equilibrium (page 58), but not necessarily static equilibrium.



In taking the algebraic sum, $\Sigma F \cos \alpha$, or $\Sigma F \cos \beta$, components acting in the directions OX and OY are positive (+), in the opposite directions negative (−). Also angles with OX and OY are measured from OX and OY around towards the left.

2d. *The algebraic sum of the moments of the forces about any point in their plane must be zero.*

Hence if p_1, p_2, p_3, etc., are the perpendiculars from any given point in the plane upon the directions of the forces F_1, F_2, F_3, etc., then

$$F_1 p_1 + F_2 p_2 + F_3 p_3 + \ldots = \Sigma F p = 0. \quad . \quad . \quad . \quad . \quad (3)$$

When this condition is complied with, there is no rotation about the point selected. But there may be rotation about some other point. In order, then, that there may be static equilibrium, both of these conditions must be complied with. We have therefore three equations of condition.

In taking the algebraic sum ΣFp of the moments of the forces, rotation counter-clockwise is taken as positive (+), and clockwise as negative (−).

COR. If *three* co-planar forces act on a rigid body at different points, and the body is in equilibrium, the line representatives of these three forces, if produced, intersect in a common point. For the resultant of any two of them must pass through their point of intersection and be equal and opposite to the third and in the same straight line with it.

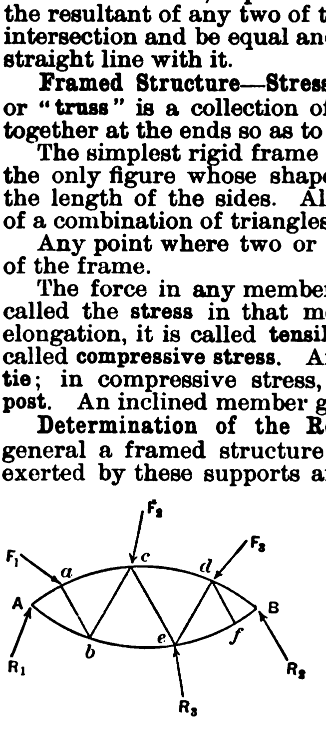
Framed Structure—Stress, etc.—A framed or jointed structure or "**truss**" is a collection of straight members pinned or jointed together at the ends so as to make a rigid frame.

The simplest rigid frame is obviously a triangle, because that is the only figure whose shape cannot be altered without changing the length of the sides. All rigid frames must consist, therefore, of a combination of triangles.

Any point where two or more members meet is called an **apex** of the frame.

The force in any member which resists change of its length is called the **stress** in that member (page 7). If the stress resists elongation, it is called **tensile stress**. If it resists shortening, it is called **compressive stress**. Any member in tensile stress is called a **tie**; in compressive stress, a **strut**. A vertical strut is called a **post**. An inclined member generally is called a **brace**.

Determination of the Reactions of a Framed Structure.—In general a framed structure rests upon supports. The pressures exerted by these supports are called the **reactions** of the supports. These reactions usually have to be determined.



Thus if the co-planar forces F_1, F_2, F_3 act at the apices a, c, d of a rigid framed structure, and if R_1, R_2, R_3 are the unknown reactions or pressures in the same plane exerted by the supports at the apices A, B, and e, then if there is equilibrium of the frame, the algebraic sum of all the vertical components must be zero; the algebraic sum of all the horizontal components must be zero; the algebraic sum of all the moments about any point in the plane of the frame must be zero.

If α_1, α_2, α_3 are the angles made by the forces F_1, F_2, F_3 with the horizontal, and a_1, a_2, a_3 the angles made by the reactions R_1, R_2, R_3 with the horizontal, we have then

$$F_1 \cos\alpha_1 + F_2 \cos\alpha_2 + F_3 \cos\alpha_3 + R_1 \cos a_1 + R_2 \cos a_2 + R_3 \cos a_3 = 0. \quad (1)$$

In this equation components towards the right are positive (+), and towards the left negative (−).

If β_1, β_2, β_3 are the angles made by the forces F_1, F_2, F_3 with the vertical, and b_1, b_2, b_3 the angles made by the reactions R_1, R_2, R_3 with the vertical, we have

$$F_1 \cos\beta_1 + F_2 \cos\beta_2 + F_3 \cos\beta_3 + R_1 \cos b_1 + R_2 \cos b_2 + R_3 \cos b_3 = 0. \quad (2)$$

In this equation components upwards are positive (+), and downwards negative (−).

Again, if we take any point, as for instance the point B, as a centre of moments, and let p_1, p_2, p_3 be the lever-arms of the forces

F_1, F_2, F_3, and L_1, L_2, L_3 be the lever-arms of the reactions, we have, since in this case $L_3 = 0$,

$$R_1L_1 + R_2L_2 + F_1p_1 + F_2p_2 + F_3p_3 = 0. \quad . \quad . \quad . \quad . \quad (3)$$

Each moment in equation (3) must be taken with its proper sign (+) for counter-clockwise rotation, and (−) for clockwise rotation.

If the directions of all the forces and reactions are known as well as their points of application, and if the forces F_1, F_2, F_3 are also known, we have then three equations between three unknown quantities, R_1, R_2, and R_3, and can therefore determine them. If there are more than three reactions unknown, we cannot determine them. There are then more unknown quantities than equations of condition.

If there are but two reactions, R_1 and R_2, that is, if R_3 then is zero, we can determine R_1 and R_2 from the equations (1) and (2).

We can also in such case determine R_1 directly from equation (3), and thus have, since $R_3 = 0$,

$$R_1L_1 + F_1p_1 + F_2p_2 + F_3p_3 = 0.$$

By taking moments about A, we can in the same way determine R_2 directly, when $R_3 = 0$.

Determination of the Stresses in a Framed Structure.—As soon as all the external forces acting upon a framed structure, including the reactions, are known we can proceed to find the stresses in the various members. We can make use of two methods. The first method is based upon the fact that the algebraic sum of vertical and horizontal components is zero. We call it the "method by resolution of forces." The second method is based upon the fact that the algebraic sum of moments is zero. We call it the "method by moments," or the "method by sections."

1. **Method by Resolution of Forces.***—Since the frame is in equilibrium there must be equilibrium at every apex of the frame. Hence all the forces acting at any apex must form a system of concurring forces in equilibrium.

But the necessary and sufficient condition for equilibrium for a system of concurring forces is that the resultant shall be zero. That is, the algebraic sum of the horizontal components of all forces acting at an apex must be zero, and the algebraic sum of all the vertical components must be zero.

Take for instance the apex a of the preceding figure (page 100). At this point we have acting the force F_1 and the stresses in the members Aa, ab, and ac. These four forces form a system of concurring forces in equilibrium.

Hence if α_1, α_2, α_3 are the angles made by the members Aa, ab and ac with the horizontal, and a_1 the angle made by F_1 with the horizontal, and we denote the stresses in the corresponding members by aA, ab, ac, we have

$$F_1 \cos a_1 + aA \cos \alpha_1 + ab \cos \alpha_2 + ac \cos \alpha_3 = 0. \quad . \quad . \quad (1)$$

If β_1, β_2, β_3 are the angles made be the members Aa, ab, and ac with the vertical, and b_1 the angle made by F_1 with the vertical,

* For corresponding graphic method see page 135.

and we denote the stresses in the corresponding members by aA, ab, ac, we have

$$F_1 \cos b_1 + aA \cos \beta_1 + ab \cos \beta_2 + ac \cos \beta_3 = 0. \quad . \quad . \quad (2)$$

Components towards the right or upwards are positive, towards the left or downwards negative. Angles are measured from the horizontal aX and vertical aY around towards the left.

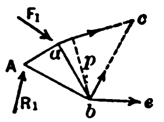
Since we have thus two equations of condition, *this method can be applied at any apex when all the forces except two are known.*

If more than two are unknown at any apex, it cannot be applied at that apex.

If the value of a stress as found by (1) and (2) comes out positive (+), it shows that the stress in the member is *away* from the apex or *tensile*. If it comes out negative, the stress is towards the apex or compressive. (See Example 2, page 104, for illustration.)

2. Method by Moments, or the "Method of Sections."*—Suppose the frame completely divided into two parts by a section cutting any member the stress in which is desired. Then the stresses which existed in the members before they were cut must evidently hold in equilibrium the external forces acting upon each of the two parts into which the frame is divided.

Thus if we wish to find the stress in any member ac (see figure, page 100), take a section cutting ac, bc and be, thus completely dividing the frame into two portions, and consider the *left-hand* portion only. Then the stresses in ac, bc and be must hold in equilibrium the external forces R_1 and F_1.



Place arrows on each of the cut pieces as in the figure, *always pointing towards the section.* Now if we take moments about the apex b, that is, if we take the point of moments at the point of intersection of the other members cut by the section, whose stresses are unknown, their moments relative to this point will be zero. We have then the algebraic sum of the moments of the external forces F_1 and R_1 and the moment of the stress in ac, all with reference to b, equal to zero. Hence, denoting the stress in ac by ac and its lever-arm by p, we have

$$ac \times p + \Sigma \text{ moments of external forces} = 0.$$

If then the external forces and their lever-arms are known and the lever-arm p of ac is known, we can find the stress ac.

The moments in the algebraic sum must be taken with their proper signs, (+) for rotation counter-clockwise, and (−) for rotation clockwise, and the moment of ac with the sign indicated by the rotation due to its arrow. Thus in our figure the moment of R_1 is negative, of F_1 negative, and of ac negative. If the stress comes out positive, it indicates, as before, that it acts away from the apex of the cut member or is tensile. If negative, towards the apex or compression. (See Example 2, page 104, for illustration.)

This method is general and can always be applied when all the cut members whose stresses are unknown, except the one whose stress is desired, meet in a point.

Thus if two of the cut pieces are parallel, their intersection is at an infinite distance.

Then if we wish to find the stress in cb, we take a section cutting ab, bc and cd. The intersection of ab and cd is at an infinite dis-

* For corresponding graphic method see page 148.

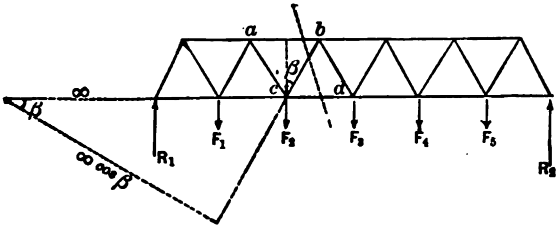
tance. We therefore have the lever-arm for cb, $\infty \cos \beta$, where β is the angle of cb with the vertical. Hence

$$R_1 \infty - F_1 \infty - F_2 \infty + cb \times \infty \cos \beta = 0,$$

or

$$cb = -(R_1 - F_1 - F_2) \sec \beta.$$

The algebraic sum of the external forces $(R_1 - F_1 - F_2)$ is called in this case the **shearing** force. For horizontal chords and vertical forces we have, then, the *stress in any brace equal to the shear multiplied by the secant of the angle which the brace makes with the*



vertical. This shear should always be taken as acting at the end c of the brace belonging to the left-hand portion. If, then, it is positive, or if R_1 is greater than $F_1 + F_2$, it acts upward at c and hence gives compression in cb. Therefore we have the minus sign in the equation above for the value of the stress in cb. (See Example 4, page 106.)

Superfluous Members.—In general the external forces acting upon a rigid frame are always known or must first be found. The stresses in the members are required. Since every apex of the frame is in equilibrium, we have at every apex a system of concurring forces in equilibrium.

We have then two equations of condition in order that the resultant shall be zero, viz.,

$$\Sigma F \cos \alpha = 0,$$
$$\Sigma F \cos \beta = 0,$$

or the algebraic sums of the horizontal and vertical components must be zero.

If, then, all the forces acting at any apex except two are known, these two can be found. But if at *every* apex there are more than two forces which are *necessarily* unknown, the problem is indeterminate, and the frame has superfluous members.

Criterion for Superfluous Members.—The simplest rigid frame is a triangle, because that is the only figure whose shape cannot change without changing the length of its sides. All rigid frames must consist therefore of a combination of triangles.

Any one member of the frame fixes the position of two apices, one at each end. Every other apex after the first two requires two members to fix its position. If then, n, is the number of apices, $2(n-2)$ will be the number of members lacking one. Let m be the number of members. Then, if there are no superfluous members, we must have

$$m = 2(n-2) + 1 = 2n - 3.$$

If m is less than $2n - 3$, there are not members enough.

If m is greater than $2n - 3$, there are superfluous members.

EXAMPLES.

(1) *In the cases of the three frames represented by Figs.* 1, 2, 3, *each supporting a weight F at the apex, show that in the first case there are not enough members and the frame is not rigid; in the second case the frame is rigid; in the third case there is a superfluous member.*

Ans. From our criterion, $m = 2n - 3$, page 103, we have the number of apices in the first case $n = 6$. Hence the number of members should be $m = 9$. But the number of members is only 8, or less than the number necessary.

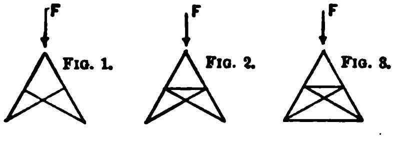


FIG. 1. FIG. 2. FIG. 3.

In the second case $n = 6$ and m should be 9, and the number of members is 9.

In the third case $n = 6$ and m should be 9, but the number of members is 10, or greater than the number necessary.

(2) *A rigid frame ABC, consisting of two rafters AB and AC and a horizontal tie BC, supports a load F at the apex A. If the angles made by the rafters with the horizontal are α_1 and α_2 at B and C, find the stresses S_1, S_2, S_3 in the rafters AB, AC and the tie BC, for equilibrium; also the pressures R_1 and R_2 of the supports. The weight of rafters and tie neglected.*

Ans. Let the pressures or reactions of the support be R_1 and R_2 at B and C, Fig. 1, and the stresses be S_1, S_2 and S_3 in the rafters AB and AC and the tie BC.

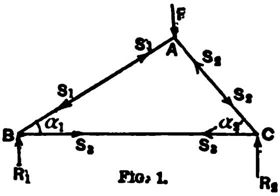


FIG. 1.

1st Method: By Resolution of Forces.—(Page 101.) The forces acting at each apex must constitute a system of forces in equilibrium.

Let us take first the apex A as origin, Fig. 2. We have here the force F and the two stresses S_1 and S_2, constituting a system of concurring forces in equilibrium. Therefore the algebraic sum of the horizontal forces must be zero and the algebraic sum of the vertical forces must be zero. Hence giving the proper signs to F and the sines and cosines of the angles α_1 and α_2 (page 102), we have

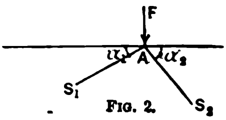


FIG. 2.

$$-S_1 \cos \alpha_1 + S_2 \cos \alpha_2 = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$-S_1 \sin \alpha_1 - S_2 \sin \alpha_2 - F = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2) we obtain

$$S_1 = -\frac{F \cos \alpha_2}{\sin (\alpha_1 + \alpha_2)}, \qquad S_2 = -\frac{F \cos \alpha_1}{\sin (\alpha_1 + \alpha_2)}. \quad . \quad . \quad (3)$$

In equations (3) the ($-$) sign denotes direction towards the origin A as indicated in Fig. (1). *A negative result then denotes compression.*

At the apex B we have the stresses S_1 and S_3 and the reaction R_1 in equilibrium. At the apex C we have S_2, S_3 and R_2 in equilibrium. Hence for the algebraic sum of the horizontal components at B we have, taking the origin at B,

$$S_3 + S_1 \cos \alpha_1 = 0,$$

and for the algebraic sum of the horizontal components at C we have, taking the origin at C,

$$-S_3 - S_2 \cos \alpha_2 = 0.$$

From both equations we have, from (3),

$$S_3 = -S_1 \cos \alpha_1 = -S_2 \cos \alpha_2 = +\frac{F \cos \alpha_1 \cos \alpha_2}{\sin(\alpha_1 + \alpha_2)}. \quad . \quad . \quad (4)$$

The positive result denotes direction away from the origin in each case, or *tension*, as shown in Fig. 1.

At the apex B we have for the algebraic sum of the vertical components

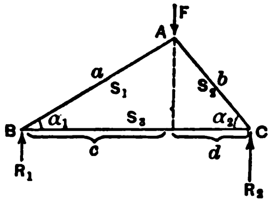
$$S_1 \sin \alpha_1 + R_1 = 0, \quad \text{or} \quad R_1 = +\frac{F \cos \alpha_2 \sin \alpha_1}{\sin(\alpha_1 + \alpha_2)}. \quad . \quad . \quad . \quad (5)$$

At the apex C we have

$$S_2 \sin \alpha_2 + R_2 = 0, \quad \text{or} \quad R_2 = +\frac{F \cos \alpha_1 \sin \alpha_2}{\sin(\alpha_1 + \alpha_2)}. \quad . \quad . \quad . \quad (6)$$

The positive result denotes upward direction for R_1 and R_2.

In all formulas the *acute values of the angles are to be used.*



2d Method: By Moments.—Let the horizontal distances of F from B and C be c and d.

Let the length of the rafters be a and b.

Then we have

$$a \cos \alpha_1 = c, \quad b \cos \alpha_2 = d, \quad b \sin \alpha_2 = a \sin \alpha_1.$$

Since all the forces acting on the frame are in equilibrium we have the algebraic sum of the horizontal and vertical external forces zero. Hence

$$R_1 + R_2 - F = 0.$$

Also taking moments about C, we have

$$-R_1(c + d) + Fd = 0, \quad \text{or} \quad R_1 = \frac{Fd}{c+d} = \frac{F \cos \alpha_2 \sin \alpha_1}{\sin(\alpha_1 + \alpha_2)};$$

and taking moments about B, we have

$$R_2(c + d) - Fc = 0, \quad \text{or} \quad R_2 = \frac{Fc}{c+d} = \frac{F \cos \alpha_1 \sin \alpha_2}{\sin(\alpha_1 + \alpha_2)}.$$

If we conceive a section through AB and BC, we have as on page 102, taking moments about C,

$$-S_1(c + d) \sin \alpha_1 - R_1(c + d) = 0, \quad \text{or} \quad S_1 = -\frac{R_1}{\sin \alpha_1} = -\frac{F \cos \alpha_2}{\sin(\alpha_1 + \alpha_2)}$$

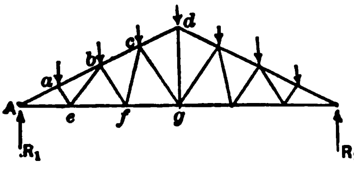
The minus sign denotes *compression*. If in the same way we cut AC and CB and take moments about B, we have

$$-S_2(c + d) \sin \alpha_2 - Fc = 0, \quad \text{or} \quad S_2 = -\frac{Fc}{(c+d)\sin \alpha_2} = -\frac{F \cos \alpha_1}{\sin(\alpha_1 + \alpha_2)}.$$

Again, cut AB and BC and take moments about A, and we have

$$S_3 \times a \sin \alpha_1 - R_1 c = 0, \quad \text{or} \quad S_3 = \frac{R_1}{\tan \alpha_1} = \frac{F \cos \alpha_1 \cos \alpha_2}{\sin(\alpha_1 + \alpha_2)}.$$

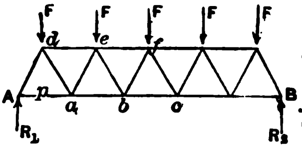
(3) *A roof-truss has a span of* 50 *ft. and a centre height of* 12.5 *ft. Each rafter is divided into four equal panels, and the lower horizontal tie is divided into six equal panels. The bracing is as shown in the figure. Find the stresses in the members by two methods, for a weight of* 800 *lbs. at each upper apex.*



Ans. $R_1 = R_2 = +2800$ lbs.

Stress in $Aa = -6260$ lbs., $ab = -5813$ lbs., $bc = -4696$ lbs.,
$cd = -3577$ lbs., $Ae = +5600$ lbs., $ef = +4802$ lbs.,
$fg = +4003$ lbs., $ae = -720$ lbs., $eb = +720$ lbs.,
$bf = -1081$ lbs., $fc = +920$ lbs., $cg = -1443$ lbs.,
$gd = +2401$ lbs. (See Example (1), page 104.)

(4) *A bridge-truss l ft. long is divided into five equal panels in the lower chord and four equal panels in the upper chord. The depth is constant and equal to d; the panel length is p. The bracing is isosceles as shown in the figure. Find the stresses for a load F lbs. at each upper apex.*



Ans. $R_1 = R_2 = +2.5F$;

$$Aa = +\frac{R_1 p}{2d}, \quad ab = \frac{1.5R_1 p - Fp}{d}, \quad bc = \frac{2.5R_1 p - 3Fp}{d},$$

$$de = -\frac{R_1 p - \frac{1}{2}Fp}{d}, \quad ef = -\frac{2R_1 p - 2Fp}{d};$$

$Ad = -R_1 \sec\beta$, $ae = -(R_1 - F)\sec\beta$, $bf = -(R_1 - 2F)\sec\beta$,
$da = +R_1 \sec\beta$, $eb = +(R_1 - F)\sec\beta$,

where β is the angle made by the braces with the vertical.

(5) *A weight of 6 lbs. hangs on the arm of a safety-valve at a distance of 18 inches from the fulcrum. The valve-spindle is attached at 1 inch from the fulcrum. Disregarding friction and the weight of the arm, find the steam pressure for equilibrium.*

Ans. 108 lbs.

(6) *In a wheel and axle the radius of the axle is r, and of the wheel R. A weight Q hangs by a rope wound about the axle. Find the force P acting tangent to the wheel in order to hold Q suspended, disregarding friction.*

Ans. $P = \dfrac{Qr}{R}$.

(7) *A shopkeeper has correct weights but an untrue balance, one arm of which is a and the other b. He serves out to each of two customers, as indicated by his balance, W lbs. of a commodity, using first one scale-pan and then the other for the commodity. Does he gain or lose?*

Ans. Loses $W\dfrac{(a-b)^2}{ab}$ lbs.

(8) *The arms of a balance are unequal, and one of the scales is loaded. A body, the true weight of which is P lbs., appears, when placed in the loaded scale, to weigh W lbs., and when placed in the other scale to weigh W' lbs. Find the ratio of the arms and the weight with which the scale is loaded.*

Ans. Ratio of arms $= \dfrac{W-P}{P-W'}$; weight required $= \dfrac{P^2 - WW'}{W-P}$.

(9) *A square and a rectangle of uniform thickness and density are joined in one plane at a common side. Find the length of the rectangle in order that the two may balance about that side, the density of the rectangle being one half of that of the square.*

Ans. The length of the rectangle = a diagonal of the square.

(10) *The inscribed circle being cut out of a right-angled triangle, the sides of which are 3, 4, 5, find the centre of mass of the remainder.*

Ans. Take side 3 as axis of X, and side 4 as axis of Y. Then

$$\bar{x} = 1, \quad \bar{y} = \frac{8 - \pi}{6 - \pi}.$$

(11) *A cubical box half filled with water is placed upon a rectangular board, so that the edges of its base are parallel to those of the board. If the board is slowly inclined to the horizon about an edge, and the box is prevented from sliding, at what angle will the box just tend to overturn?*

Ans. 45°.

(12) *Let the forces* $+4, -7, +8, -3$ *lbs. act perpendicularly to a straight line at points* A, B, C *and* D, *so that* $AB = 5$ *ft.,* $BC = 4$ *ft.,* $CD = 2$ *ft. Find the resultant and its point of application* E.

Ans. $R = 2$ lbs., $AE = 2$ ft.

(13) *Let three forces which, if concurring, would be in equilibrium act each in the side of a triangle which represents them in magnitude and direction. If not concurring, show that they are equivalent to a couple whose moment is proportional to the area of the triangle.*

(14) *Three forces act at the middle points of the sides of a rigid triangular plate in its plane, each force being perpendicular and proportional to the side on which it acts. If the forces are all inward or outward, show that the resultant is zero.*

(15) *A system of any number of co-planar forces being represented in magnitude and direction by the sides of a closed polygon taken the same way round, show that the sum of their moments about any point in their plane is constant and independent of the position of the point.*

false (16) *Forces of* 10, 20, 30 *and* 40 *pounds act on a rigid body at* A, B, C, D, *the four corners of a square whose side is* 2 *ft. and in its plane. Their inclinations to* AB, BC, CD, DA *are* 45°, 90°, 30°, 60° *respectively. Show that the resultant is a force of* 35.65 *lbs., and that its line of action is distant* 3.03 *ft. from* C.

(17) *Parallel forces in the same direction, and of the magnitudes* 10, 15, 20, 25 *lbs., act at points* A, B, C, D *respectively of a straight rod, the distances* AB, BC, CD *being* 2, 3, 4 *ft. respectively. Find the distance of the point of application from* A.

Ans. 5.07 feet.

(18) *Two parallel forces in opposite directions of* 20 *and* 5 *lbs. act at points* A *and* B *of a rigid body* 4 *ft. apart. Find the distances from* A *and* B *of the point in which their resultant line of action cuts* AB.

Ans. $1\frac{1}{3}$ and $5\frac{1}{3}$ ft.

(19) *The numerical measures of the magnitude of a force which acts upon a point in a given direction, and of the co-ordinates of the*

point in the plane of the force, are denoted by a, b, c; but it is not known which is which. Find the centre of all the forces which may be represented.

Ans. $\bar{x} = \bar{y} = \dfrac{ab + bc + ca}{a + b + c}$.

(20) *Forces* 1, -3, -5, 7 *act on a rigid rod at points* A, B, C, D, *whose distances are such that* $AB = 3$, $BC = 2$, $CD = 2$. *Find the resultant.*

Ans. A couple whose moment is 15 units.

(21) *Three equal and co-directional forces* (F) *act at three corners of a square* (*side* $= a$) *perpendicularly to the square. Find the magnitude of the force which, applied at the other corner of the square, would with the given forces constitute a couple, and the moment of the couple.*

Ans. $3F$; $2aF\sqrt{2}$.

(22) ABC *is a triangle right-angled at* B. *At* A *a force* F *is applied in the plane of the triangle perpendicular to* AC; *at* C *a force* $2F$ *in the same direction; at* B *a force* $3F$ *in the opposite direction. Find the moment of the resulting couple.*

Ans. $\dfrac{F(AB^2 - 2BC^2)}{AC}$.

(23) *Two forces* P *and* Q *act at the ends* A *and* B *of a straight lever* AB *without mass. To find the position of the fulcrum in order that equilibrium may be produced, the inclination of* P *and* Q *with* AB *being* α *and* β.

Ans. Let $AB = c$, and x, y the distances of the fulcrum from A and B respectively. Then

$$x = \frac{Qc \sin \beta}{P \sin \alpha + Q \sin \beta}, \quad y = \frac{Pc \sin \alpha}{P \sin \alpha + P \sin \beta}.$$

(24) *A rod* CD, *without mass, moving about a smooth hinge at* C, *presses at* D *against a wall inclined at an angle* α *with the horizon, and has a weight* W *suspended at its centre. Find the inclination* θ *of the rod to the horizon in order that the pressure at* D *may be* $\frac{1}{2}W$.

Ans. $\theta = \frac{1}{2}\alpha$.

(25) *Two weights* P *and* Q *are suspended from the extremities of a lever without mass, in the form of a circular arc, which rests with its convexity downwards upon a horizontal plane. If* 2α *is the central angle of the arc and* θ *the central angle from the point of attachment of* P *to the point of tangency with the horizontal plane, find* θ *for equilibrium.*

Ans. $\tan \theta = \dfrac{P - Q}{P + Q} \cdot \tan \alpha$.

(26) *The arms of a balance are unequal, and a substance placed successively in each scale appears to weigh* P *and* Q *lbs. Show that the lengths of the arms, disregarding the mass of the balance, are as* $\sqrt{P}$ *to* $\sqrt{Q}$.

(27) *If weights* P *and* Q, P *being the greater, balance on a lever* ACB *without mass, about a fulcrum at* C, *and the weights are inter-*

changed, show that the additional weight required at A for equilibrium will be

$$\frac{P^2 - Q^2}{Q}.$$

(28) *It is found that a body weighs P when suspended at the end A of a balance without mass, and Q when suspended at B. Show that the fulcrum ought to be shifted towards A a distance equal to*

$$\frac{\sqrt{P} - \sqrt{Q}}{\sqrt{P} + \sqrt{Q}} \cdot \frac{AB}{2}.$$

(29) *The length of a false balance-beam is 3 ft. A body in one scale weighs 4 lbs.; in the other, 6 lbs. 4 oz. Find the true weight of the body and the lengths of the lever-arms.*

Ans. True weight = 5 lbs.; lengths of arms, 1 ft. 4 in. and 1 ft. 8 in.

(30) *Three uniform rods AB, BC, CD, rigidly connected so as to form three sides of a square, rest upon a fulcrum at A. Suppose the weight of each rod to act at its centre. Find the inclination θ of AB with the horizon.*

Ans. $\tan\theta = \frac{4}{3}$.

(31) *AB, CD, DE are three equal uniform rods, rigidly connected at right angles, B being the middle point of CD. Suppose the weight of each rod to act at its centre, and the system to hang from a fulcrum at A. Find the inclination θ of AB to the horizon for equilibrium.*

Ans. $\tan\theta = 6$.

CHAPTER V.

EQUILIBRIUM OF A PERFECTLY FLEXIBLE INEXTENSIBLE STRING.

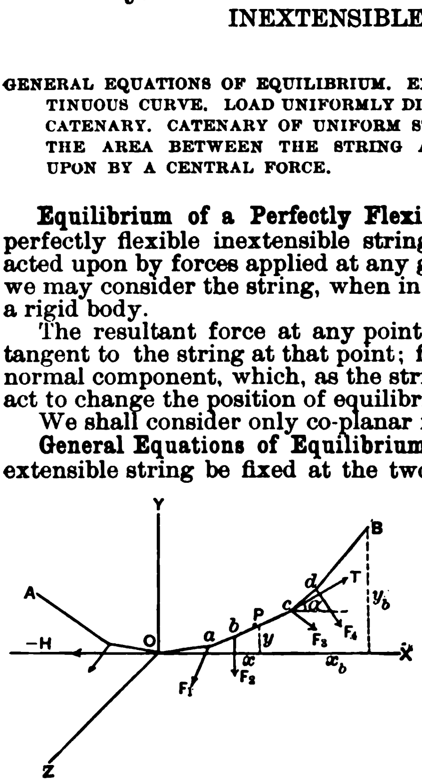
GENERAL EQUATIONS OF EQUILIBRIUM. EXTERNAL FORCES VERTICAL. CONTINUOUS CURVE. LOAD UNIFORMLY DISTRIBUTED OVER THE HORIZONTAL. CATENARY. CATENARY OF UNIFORM STRENGTH. LOAD PROPORTIONAL TO THE AREA BETWEEN THE STRING AND HORIZONTAL. STRING ACTED UPON BY A CENTRAL FORCE.

Equilibrium of a Perfectly Flexible Inextensible String.—If a perfectly flexible inextensible string is fixed at two points and acted upon by forces applied at any given points in any directions, we may consider the string, when in its position of equilibrium, as a rigid body.

The resultant force at any point must then act in a direction tangent to the string at that point; for otherwise there would be a normal component, which, as the string is perfectly flexible, would act to change the position of equilibrium of that point.

We shall consider only co-planar forces.

General Equations of Equilibrium.—Let a perfectly flexible inextensible string be fixed at the two points A and B and be acted upon by external forces in its plane. It is required to determine the tension T of the string at any point P, and the position of any point P for equilibrium, disregarding the weight of the string.



The string when in equilibrium will evidently take the form of a polygon, if the forces are applied at points or are "discontinuous"; the tension in any segment, as bc, being the resultant of the tension in the preceding segment ab and the force F_2 at b.

Take the origin of co ordinates at the lowest point O of the string, and let the co-ordinates of any point P of the string be x and y. Let the external forces acting upon the portion OP of the string be F_1, F_2, etc.; the co-ordinates of their points of application a, b, etc., be given by (x_1, y_1), (x_2, y_2); etc.; and their angles with the axes of X and Y be given by (α_1, β_1), (α_2, β_2), etc.

Then the algebraic sum of the horizontal and vertical components of the external forces between O and P is

$$F_x = F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + \ldots = \sum_P^0 F \cos \alpha; \quad . \quad . \quad . \quad (1)$$

$$F_y = F_1 \cos \beta_1 + F_2 \cos \beta_2 + \ldots = \sum_P^0 F \cos \beta. \quad . \quad . \quad . \quad (2)$$

Also the algebraic sum of the moments of all the external forces between O and P with reference to O, or the moment about the axis of Z, is

$$M_z = \sum_P^0 Fx \cos \beta - \sum_P^0 Fy \cos \alpha. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In taking the algebraic sums, components to the right or upward are positive, to the left or downwards negative. Also rotation counter-clockwise is positive, and clockwise negative.

Let the tension at the point P be T, making the angles α and β with the axes of X and Y, and let the horizontal tension at the lowest point O be H.

If the portion of the string from O to P is in equilibrium, we can treat it as rigid, and we have then the algebraic sum of the horizontal and vertical components of all the forces acting upon it equal to zero; also the algebraic sum of the moments of all the forces acting upon it, with reference to any point as O, equal to zero.

Hence the conditions for equilibrium are

$$\left.\begin{aligned} &-H + F_x + T \cos \alpha = 0; \\ &F_y + T \cos \beta = 0; \\ &M_z + Tx \cos \beta - Ty \cos \alpha = 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

We have also

$$\cos^2 \alpha + \cos^2 \beta = 1.$$

We have then four equations between the four qnantities H, T, α and β, and can therefore find them for any given x and y. Equations (4) are general and apply whether the forces are discontinuous or applied continuously along the string.

External Forces Vertical.—If all the external forces acting upon the string are vertical, we have $F_x = 0$ and $F_y = \Sigma_P^0 F$. Hence from equations (4) of the preceding Article,

$$T \cos \alpha = H;$$

$$T \cos \beta = - \Sigma_P^0 F.$$

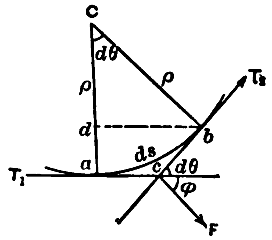
That is, for a perfectly flexible inextensible string in equilibrium under the action of vertical external forces, whether the forces are applied continuously along the string or discontinuously:

1st. *The horizontal component of the tension at any point is constant and equal to the horizontal tension at the lowest point.*

2d. *The vertical component of the tension at any point is equal to the algebraic sum of all the forces between that point and the lowest point.*

Continuous Curve—Tangential and Normal Components.—If the forces are applied continuously along the string, then the shape of the string when in equilibrium will be a continuous curve instead of a polygon.

Let $ab = ds$ be the length of an indefinitely small portion of the curve. Let the resultant force in any direction continuously applied over ds be F, so that the force per unit of length is $\frac{ds}{F}$. Let the tension of the string at a be T_1, tangent to the curve at a, and the tension at b be T_2, tangent to the curve at b. Let the very small angle between these tangents be $d\theta$, and let the force F make the angle ϕ with the tangent at a.



Then since for equilibrium we may consider ab as rigid, the three co-planar forces T_1, T_2 and F are in equilibrium and must intersect at a common point c (page 85).

We can consider them, then, as three forces concurring at c and in equilibrium. If then we resolve these forces along the tangent at a, we have

$$T_2 \cos d\theta + F \cos \phi - T_1 = 0.$$

When $ab = ds$ is indefinitely small, the points a and b come together, $d\theta$ becomes zero, and $\cos d\theta = 1$. Hence

$$-\frac{F}{ds} \cos \phi = \frac{T_2 - T_1}{ds} = \frac{dT}{ds}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

That is, *the tangential component of the external force per unit of length at any point is equal to the variation of tension per unit of length at that point.*

Again, resolving the forces along the normal at a, we have

$$T_2 \sin d\theta - F \sin \phi = 0.$$

If ρ is the radius of curvature, we have $bd = \rho \sin d\theta$. When ds is indefinitely small, we can take $bd = ds = ab$. Hence $\sin d\theta = \frac{ds}{\rho}$. Substituting this, we have, when the points a and b come together,

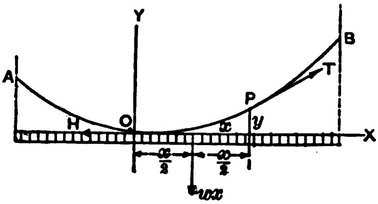
$$\frac{F}{ds} \sin \phi = \frac{T_2}{\rho}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

That is, *the normal component of the external force per unit of length at any point is equal to the tension at that point divided by the radius of curvature at that point.*

COR. If the external force per unit of length at every point of the string is normal to the string, $\phi = 90°$, and, from equation (1), $T_2 - T_1 = 0$ or $T_1 = T_2$ at every point. That is, *the tension is constant throughout the string.* This is the case when the string is stretched over any smooth surface whose pressure on the string at every point is normal, and acted upon by no forces except the normal pressure of the surface and two equal terminal tensions. In such case $u = \frac{T}{\rho}$, or the normal pressure of the surface per unit of length at any point is inversely proportional to the radius of curvature at that point. That is, $u\rho = T =$ the constant tension in the string.

Load Uniformly Distributed over the Horizontal Projection of the String.—This is approximately the case of the ordinary suspension bridge.

Let the mass of the unit load or load per unit of horizontal projection be constant and equal to w in gravitation units (page 6) or wg in absolute units. Let H be the horizontal tension at the lowest point O, and T be the tension at any point P of the string, both in gravitation units.



Equation of the Curve.—Let x and y be the co-ordinates of any point P of the string, the origin being taken at the lowest point O. Then we can consider any portion of the string OP when in equilibrium as rigid and acted upon by the forces H, T, and the entire load wx between O and P. The resultant force wx of the load between O and P acts at the centre of mass of the load, or, since the load is uniformly distributed, half way between O and P. If then we take moments about P, we have for the moment of the load with reference to P,

$$wx \times \frac{x}{2} = \frac{wx^2}{2}.$$

We have then for equilibrium

$$\frac{wx^2}{2} - Hy = 0, \quad \text{or} \quad x^2 = \frac{2H}{w}y. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The curve of the string is then *a parabola* whose axis is vertical and whose parameter is $\frac{2H}{w}$. If w is constant and the parameter is constant, H is constant. Hence, *the tension at the lowest point is constant for all parabolas having the same parameter, when the load per unit of horizontal projection is constant, whatever may be the length of the curve.*

Tension at the Lowest Point.—To find the tension H at the lowest point, we have only to substitute in equation (1) the co-ordinates of some known point. Thus let x_b and y_b be the co-ordinates of the end B. Then equation (1) gives

$$H = \frac{wx_b^2}{2y_b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Or we may find this value of H directly by taking moments about B. Thus the resultant of the load between the lowest point O and B is wx_b and it acts at the centre of mass of the load, or, since the load is uniformly distributed, half way between the lowest point O and B. If then we take moments about B, the moment of the load is $wx_b \times \frac{x_b}{2} = \frac{wx_b^2}{2}$. We have then for equilibrium

$$\frac{wx_b^2}{2} - Hy_b = 0, \quad \text{or} \quad H = \frac{wx_b^2}{2y_b}.$$

Slope of the Curve.—For the slope or inclination α of the curve at any point with the horizontal, we have seen already, page 111,

that for vertical forces the horizontal component of the tension at any point is constant and equal to H, and the vertical component is wx. We have then for the slope at any point P

$$\tan\alpha = \frac{wx}{H}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For the slope at the end B we have then

$$\tan\alpha_b = \frac{2y_b}{x_b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Tension at Any Point.—For the tension T at any point P we have then

$$T = H\sqrt{1 + \left(\frac{wx}{H}\right)^2} = H\sec\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

For the tension at the end B we have

$$T_b = \frac{wx_b}{2y_b}\sqrt{x_b^2 + 4y_b^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

[**Solution of Preceding Case by Calculus.**]—Let the unit load or load per unit of horizontal projection be constant and equal to w in gravitation units (page 6).

Then referring to our general equations (4), page 111, we have in gravitation units $F_y = -wx$, $F_x = 0$, $\cos\alpha = \frac{dx}{ds}$, $\cos\beta = \frac{dy}{ds}$, where ds is the length of an element of the curve and dx, dy its horizontal and vertical projections. Therefore from equations (4), page 111,

$$-H + T\frac{dx}{ds} = 0,$$

$$-wx + T\frac{dy}{ds} = 0,$$

where H and T are to be taken in gravitation units if w is taken in gravitation units.

Eliminating T, we obtain

$$Hdy = wxdx.$$

Integrating, and taking the origin at the lowest point O, so that when $x = 0$, y is also zero, we have

$$Hy = \frac{wx^2}{2}, \quad \text{or} \quad x^2 = \frac{2H}{w}y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is the equation of the curve as already found, page 113.

If we substitute the co-ordinates of the end B, x_b and y_b, in place of x and y, we have from (1), for the tension H at the lowest point,

$$H = \frac{wx_b^2}{2y_b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For the slope or inclination α of the curve at any point we have, by differentiating (1),

$$\tan\alpha = \frac{dy}{dx} = \frac{wx}{H}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The slope at the end B is then

$$\tan \alpha_b = \frac{2y_b}{x_b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

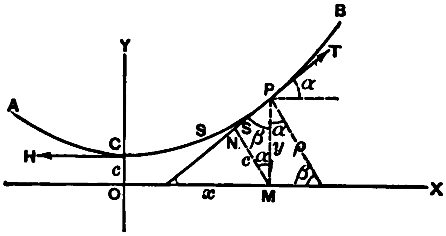
For the tension T at any point p we have

$$T = H\frac{ds}{dx} = H\frac{\sqrt{dx^2+dy^2}}{dx} = H\sqrt{1+\frac{dy^2}{dx^2}} = H\sqrt{1+\left(\frac{wx}{H}\right)^2} = H \sec \alpha \quad .(5)$$

For the tension at the end B,

$$T_b = \frac{wx_b}{2y_b}\sqrt{x_b^2 + 4y_b^2}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

[**Load Uniformly Distributed over the String.**]—The curve of equilibrium assumed under the action of gravity, by a perfectly flexible string of uniform normal section and density, when suspended from two points not in the same vertical, is called the **catenary**. In such case the load is the weight of the string and is uniformly distributed over the curve. If the unit load or weight of a unit length of the string is not constant, but varies continuously according to any law, the curve of equilibrium is called a **catenarian curve.**



Let w be the mass of the unit load, or the load per unit of length of the string, in gravitation units (page 6). Then if δ is the uniform density of the string, or the mass per unit of volume, A the constant area of normal section, and s the length of any portion of the string, the mass of that portion is δAs, and the mass per unit of length, or the load per unit of length in gravitation units, is

$$w = \delta A. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In absolute units we have $w = \delta Ag$.

Referring to our general equations (4), page 111, we have in gravitation units $F_y = -ws$, where s is the length of the string from the lowest point C to any point P. We also have $F_x = 0$, $\cos \alpha = \frac{dx}{ds}$, $\cos \beta = \frac{dy}{ds}$, where ds is the length of an element of the string and dx, dy its horizontal and vertical projections.

Hence from equations (4), page 111,

$$-H + T\frac{dx}{ds} = 0;$$

$$-ws + T\frac{dy}{dx} = 0;$$

where H and T are to be taken in gravitation units if w is taken in gravitation units.

Eliminating T, we have for the slope α at any point P

$$\tan \alpha = \frac{dy}{dx} = \frac{w}{H}s. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let $H = wc$, or $c = \frac{H}{w}$, where c is then *the length of that portion of the string whose weight is equal to the tension H at the lowest point C.* Then

$$\frac{dy}{dx} = \frac{s}{c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Differentiating (3), substituting $ds = \sqrt{dx^2 + dy^2}$, and reducing,

$$d\left(\frac{dy}{dx}\right) = \frac{ds}{c} = \frac{dx}{c}\sqrt{1 + \frac{dy^2}{dx^2}},$$

or

$$\frac{dx}{c} = \frac{d\left(\frac{dy}{dx}\right)}{\sqrt{1 + \frac{dy^2}{dx^2}}}.$$

Integrating this, we have

$$\frac{x}{c} = \log \operatorname{nat}\left[\frac{dy}{dx} + \sqrt{1 + \frac{dy^2}{dx^2}}\right] + \text{const.}$$

If we take the axis of Y passing through the lowest point C, we have $\frac{dy}{dx} = 0$, where $x = 0$. Therefore const. $= 0$ and

$$\frac{x}{c} = \log \operatorname{nat}\left[\frac{dy}{dx} + \sqrt{1 + \frac{dy^2}{dx^2}}\right] = \log \operatorname{nat}\left[\frac{s}{c} + \sqrt{1 + \frac{s^2}{c^2}}\right]. \quad . \quad . \quad (4)$$

Or, if $e = 2.718282$ is the base of the Naperian system of logarithms,

$$e^{\frac{x}{c}} = \frac{dy}{dx} + \sqrt{1 + \frac{dy^2}{dx^2}} = \frac{dy}{dx} + \frac{ds}{dx} = \frac{s}{c} + \sqrt{1 + \frac{s^2}{c^2}}. \quad . \quad . \quad (5)$$

or

$$1 + \frac{dy^2}{dx^2} = \left[e^{\frac{x}{c}} - \frac{dy}{dx}\right]^2.$$

Solving this equation, we have for the slope α at any point (see (3))

$$\tan \alpha = \frac{dy}{dx} = \frac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) = \frac{s}{c}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Integrating (6), we obtain

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) + \text{const.}$$

Now, taking the origin O (see figure) at a distance equal to $CO = c$ below the lowest point C, we have $y = c$ when $x = 0$. This gives const. $= 0$. The horizontal line OX at the distance $c = \frac{H}{w}$ below the lowest point C is called the *directrix*. The distance $OC = c = \frac{H}{w}$ is called the *parameter*.

We have then for the equation of the curve, taking the origin O at the distance $CO = c = \frac{H}{w}$ below the lowest point C,

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The point O at the distance $CO = c = \frac{H}{w}$ below the lowest point C is called *the origin* of the catenary, and equation (7) is the equation of the catenary referred to this origin.

We have from (6),

$$s = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Equation (8) gives the length of the curve from the lowest point C to any point P.

From (7) and (8) we have

$$y^2 = s^2 + c^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and differentiating (9),

$$s = y\frac{dy}{ds} = y \cos \beta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Let PM and PT be the ordinate and tangent at P, and let fall the perpendicular MN on PT. Then

$$PN = y \cos \beta = s; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and since $y^2 - s^2 = c^2$, we have

$$MN = c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Hence, given the catenary, we can construct its origin and direction as follows:

On the tangent at any point P measure off PN equal to the arc CP. At N erect a perpendicular NM to the tangent meeting the ordinate of P in M. Then the horizontal line through M is the directrix.

We have seen (page 111) that for vertical external forces the horizontal projection of the tension at any point is constant and equal to H, and the vertical component is ws. Therefore the tension T at any point is

$$T = \sqrt{H^2 + w^2s^2} = H\sqrt{1 + \frac{s^2}{c^2}}. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

But, from (9), $c^2 + s^2 = y^2$; therefore, since $w = \frac{H}{c}$,

$$T = \frac{H}{c}y = wy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

That is, *the tension at any point of the catenary is equal to the weight of a portion of the string whose length is equal to the ordinate of that point.*

From page 112 we have $w \sin \phi = \frac{T}{\rho}$, where ρ is the radius of curvature. In the present case $\phi = \beta =$ angle made by vertical with the tangent at P. Substituting $T = wy$, we have

$$\rho \sin \beta = \rho \cos \alpha = y.$$

We see then from the figure (page 115) *that the length of the radius of curvature at any point is equal to the length of the normal between that point and the directrix.*

We also see from the figure that $y \cos \alpha = c$, or $y = \dfrac{c}{\cos \alpha}$. Therefore

$$\rho = \frac{c}{\cos^2 \alpha} = c \sin^2 \alpha. \quad . \quad . \quad . \quad \quad . \quad . \quad . \quad (15)$$

We also have

$$c = s \tan \beta, \quad \text{or} \quad \frac{s}{c} = \frac{\cos \beta}{\sin \beta}.$$

Hence from equation (3), after reduction,

$$x = s \tan \beta \log \text{nat} \cot \frac{1}{2} \beta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

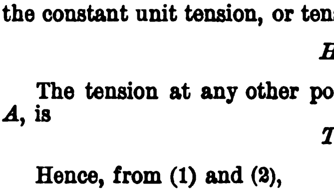
The catenary possesses other interesting properties, among which are the following:

The centre of mass of the catenary is lower than for any other curve of the same length joining the same two points.

If a common parabola is rolled on a straight line, its focus describes a catenary whose parameter c is equal to the focal distance of the parabola.

If an indefinite number of strings (without weight) are hung from the catenary, so that their lower ends are in a horizontal line and then the catenary is drawn out into a straight line, the lower ends of the strings will be in the arc of a parabola.

[**Catenary of Uniform Strength.**]—If the area of the normal section of the string at every point is proportional to the tension at that point, the *unit tension*, or tension per unit of area, will be the same at all points, and the curve assumed under the action of gravity by such a string of uniform density and perfectly flexible is called the **catenary of uniform strength.**



Let A_0 be the area of normal section of the string at its lowest point C, where the horizontal tension is H, and let t be the constant unit tension, or tension per unit of area. Then

$$H = tA_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The tension at any other point P, where the area of normal section is A, is

$$T = tA. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence, from (1) and (2),

$$A : A_0 :: T : H, \quad \text{or} \quad A = A_0 \frac{T}{H}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let δ be the uniform density of the string. Then the mass of an element of the string of length ds, or the weight in gravitation units (page 6), is $\delta A ds$. The weight in absolute units is $\delta A ds \times g$.

Referring to our general equations (4), page 111, we have for the weight in gravitation units of the string from the lowest point C to any point P

$$F_y = -\int_0^s \delta A ds.$$

We have also $F_x = 0$, $\cos\alpha = \frac{dx}{ds}$, $\cos\beta = \frac{dy}{ds}$, where ds is the length of an element of the string, and dx, dy its horizontal and vertical components. Hence, from equations (4), page 111,

$$-H + T\frac{dx}{ds} = 0;$$

$$-\int_0^s \delta A ds + T\frac{dy}{ds} = 0;$$

where H and T are to be taken in gravitation units.

Eliminating T, we have

$$\frac{dy}{dx} = \frac{\int_0^s \delta A ds}{H}, \quad \text{or} \quad d\left(\frac{dy}{dx}\right) = \frac{\delta A ds}{H} = \frac{\delta A ds^2}{T dx}.$$

Inserting the value of $A = \frac{A_0 T}{H}$, we have

$$d\left(\frac{dy}{dx}\right) = \frac{\delta A_0 ds^2}{H dx}.$$

Let $ds^2 = dx^2 + dy^2$, and let

$$\frac{\delta A_0}{H} = \frac{1}{c}, \quad \text{or} \quad c = \frac{H}{\delta A_0} = \frac{t}{\delta}. \qquad \ldots\ldots \quad (4)$$

That is, c *is the length of a string of constant cross-section* A_0 *equal to the cross-section at the lowest point* C, *and the same uniform density* δ *as the curve, whose weight is equal to the horizontal tension* H *at the lowest point*. Then

$$d\left(\frac{dy}{dx}\right) = \frac{ds^2}{c dx} = \frac{dx^2 + dy^2}{c dx},$$

or

$$\frac{d^2y}{dx^2} = \frac{1}{c}\left(1 + \frac{dy^2}{dx^2}\right),$$

or

$$\frac{\frac{d^2y}{dx^2}}{1 + \frac{dy^2}{dx^2}} = \frac{1}{c}.$$

Integrating this, we obtain

$$\tan^{-1}\left(\frac{dy}{dx}\right) = \frac{x}{c} + \text{Const.}$$

Let the axis of Y pass through the lowest point C of the curve. Then, when $x = 0$, we have

$$\frac{dy}{dx} = 0 \quad \text{and} \quad \text{Const.} = 0.$$

Hence

$$\tan\alpha = \frac{dy}{dx} = \tan\frac{x}{c}. \qquad \ldots\ldots \quad (5)$$

Integrating again, we have

$$y = -c \log \text{nat} \cos \frac{x}{c} + \text{Const.}$$

If we take the origin at the lowest point C, then, when $x = 0$, we have $y = 0$ and Const. $= 0$. Hence

$$y = -c \log \text{nat} \cos \frac{x}{c} = c \log \text{nat} \sec \frac{x}{c} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Equation (6) is the equation of the catenary of uniform strength.

From equation (5) we have

$$\alpha = \frac{x}{c} \quad \text{and} \quad d\alpha = \frac{dx}{c}.$$

If ρ is the radius of curvature, we have $\rho d\alpha = ds$, and hence

$$\rho = \frac{ds}{d\alpha} = c\frac{ds}{dx} = c \sec \frac{x}{c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If we integrate the equation $\frac{dx}{ds} = \cos \alpha = \cos \frac{x}{c}$, or $ds = \sec \frac{x}{c} dx$, we have, since, when $x = 0$, $s = 0$ and the Const. of integration is zero,

$$s = c \log n \tan \left(45° + \frac{x}{2c}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Equation (8) gives the length of the curve from the lowest point C to any point P.

From (8) we have

$$e^{\frac{s}{c}} = \tan \left(45° + \frac{x}{2c}\right) = \frac{1 + \sin \frac{x}{c}}{\cos \frac{x}{c}},$$

where $e = 2.718282$ is the base of the Naperian system of logarithms.

If we substitute $\sin \frac{x}{c} = \sqrt{1 - \cos^2 \frac{x}{c}}$ and reduce, we obtain

$$\frac{1}{\cos \frac{x}{c}} = \sec \frac{x}{c} = \frac{1}{2}\left(e^{\frac{s}{c}} + e^{-\frac{s}{c}}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

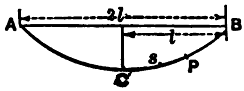
Substituting (9) in (7), we have

$$\rho = \frac{c}{2}\left(e^{\frac{s}{c}} + e^{-\frac{s}{c}}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

We have seen (page 111) that for vertical external forces the horizontal projection of the tension at any point is constant and equal to H, and the vertical component is therefore $H \tan \alpha = H \tan \frac{x}{c}$. We have then for the tension at any point P

$$T = H\sqrt{1 + \frac{dy^2}{dx^2}} = H\sqrt{1 + \tan^2 \frac{x}{c}} = H \sec \frac{x}{c}. \quad . \quad . \quad . \quad (11)$$

Let the two points of support A and B lie in a horizontal line AB. Then the curve will be symmetrical with respect to the lowest point C. Let the entire length of span AB be $2l$, then the weight of the entire string W will be given by



$$W = 2H \tan \frac{l}{c},$$

or, since, by equation (4), $c = \frac{t}{\delta}$,

$$W = 2H \tan \frac{\delta l}{t}, \quad \text{or} \quad H = \frac{W}{2} \cot \frac{\delta l}{t}.$$

The area of normal section at any point P is then, from (2) and (11),

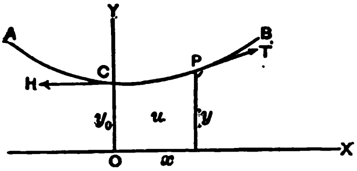
$$A = \frac{T}{t} = \frac{H}{t} \sec \frac{x}{c} = \frac{W}{2t} \cot \frac{\delta l}{t} \sec \frac{x}{c}.$$

Substituting the value of $\sec \frac{x}{c}$ from (9) and putting $c = \frac{t}{\delta}$, we have for the area of cross-section A at any point P at a distance measured along the curve from the lowest point C equal to $s = CP$,

$$A = \frac{W}{4t}\left(e^{\frac{\delta s}{t}} + e^{-\frac{\delta s}{t}}\right) \cot \frac{\delta l}{t}. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

From equation (12), if the points of support are on a horizontal, and the span AB, the weight of the entire string, its density and the unit tension are given, we can find the area of normal section at any point P at a distance s along the curve from the lowest point C.

[**Load Proportional to the Area between the String and a Horizontal.**]—Let the load on any portion of the string CP be proportional to the area $OCPx$ between the curve and a horizontal line OX. Take the origin at O in the vertical through the lowest point C, and let the distance $OC = y_0$.



Let w be the mass, or weight in gravitation units (page 6), of one unit of area of the load area between the curve and OX.

Let $H = wc^2$, or

$$\frac{H}{w} = c^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

that is, c^2 *is the area of that portion of the load area whose weight is equal to the tension H at the lowest point C.*

Let the area $OCPx$ be denoted by u. We have then for the load from C to any point P,

$$F_y = -wu = -w\int_0^x y\,dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Referring to our general equations (4), page 111, we have also $F_x = 0$, $\cos \alpha = \frac{dx}{ds}$, $\cos \beta = \frac{dy}{ds}$, and

$$-H + T\frac{dx}{ds} = 0;$$

$$-w\int_0^x y\,dx + T\frac{dy}{dz} = 0.$$

Eliminating T, we have

$$\frac{dy}{dx} = \frac{d^2u}{dx^2} = \frac{F_y}{H} = \frac{u}{c^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Multiplying by $2du$,

$$\frac{2du\,d^2u}{dx^2} = \frac{2u\,du}{c^2}.$$

Integrating,

$$\frac{du^2}{dx^2} = \frac{u^2}{c^2} + \text{Const.}$$

Now $u = \int_0^x y\,dx$, and $du = y\,dx$, or $\frac{du}{dx} = y$. Therefore, when $u = 0$, $\frac{du}{dx}$ will be equal to $y_0 = OC$, and Const. $= y_0$. Hence

$$\frac{du^2}{dx^2} = \frac{u^2}{c^2} + y_0^2, \quad \text{or} \quad dx = \frac{du}{\sqrt{\frac{u^2}{c^2} + y_0^2}}.$$

Integrating,

$$x = c \operatorname{logn} \left[\frac{u}{c} + \sqrt{\frac{u^2}{c^2} + y_0^2} \right] + \text{Const.}$$

When $u = 0$, we have $x = 0$, and Const. $= -c \operatorname{logn} y_0$. Hence

$$\frac{x}{c} = \operatorname{logn} \left[\frac{u}{cy_0} + \sqrt{\frac{u^2}{c^2 y_0^2} + 1} \right]. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Or, if $e = 2.718282$ is the base of the Naperian system of logarithms,

$$e^{\frac{x}{c}} = \frac{u}{cy_0} + \sqrt{\frac{u^2}{c^2 y_0^2} + 1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Solving this for u, we obtain

$$\text{area} = u = \frac{cy_0}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Also, since $y = \frac{du}{dx} = \sqrt{\frac{u^2}{c^2} + y_0^2}$, ... (7)

$$y = \frac{y_0}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

We have from (3) also

$$\tan\alpha = \frac{dy}{dx} = \frac{u}{c^2} = \frac{y_0}{2c}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For the tension T at any point P, since $F_y = H\frac{u}{c^2} = H\frac{dy}{dx}$,

$$T = \sqrt{F_y^2 + H^2} = H\sqrt{1 + \frac{dy^2}{dx^2}} = H \sec\alpha. \quad . \quad . \quad . \quad . \quad (10)$$

The length c is the *parameter* of the curve. From (6) we have

$$\frac{y}{y_0} = \sqrt{\frac{u^2}{c^2 y_0^2} + 1}, \quad \text{and} \quad \frac{u}{cy_0} = \sqrt{\frac{y^2}{y_0^2} - 1}.$$

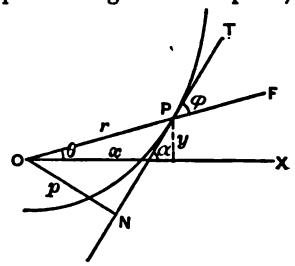
Therefore, from (4),

$$x = c \operatorname{logn} \left(\frac{y}{y_0} + \sqrt{\frac{y^2}{y_0^2} - 1} \right), \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and hence

$$c = \frac{x}{\operatorname{logn} \left(\frac{y}{y_0} + \sqrt{\frac{y^2}{y_0^2} - 1} \right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

String Acted on by Central Force.—When the lines of action of the forces applied to the elements of the string all pass through the same point, the force acting on the string is said to be **central**, and this point is the **centre of force.**



Let P be any point of the curve, and take the centre of force O as the origin, and let the radius vector $OP = r$ make the angle θ with the axis of X. Then we have

$$\cos \theta = \frac{x}{r}, \quad \sin \theta = \frac{y}{r}. \quad . \quad . \quad (1)$$

Let the force F upon the element ds at any point P make the angle ϕ with the tangent at P, and let the tangent make the angle α with the axis of x. Then

$$\cos \alpha = \frac{dx}{ds}, \quad \sin \alpha = \frac{dy}{ds}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\left. \begin{aligned} \cos \phi &= \cos (\alpha - \theta) = \cos \alpha \cos \theta + \sin \alpha \sin \theta = \frac{x}{r}\frac{dx}{ds} + \frac{y}{r}\frac{dy}{ds}; \\ \sin \phi &= \sin (\alpha - \theta) = \sin \alpha \cos \theta - \cos \alpha \sin \theta = \frac{x}{r}\frac{dy}{ds} - \frac{y}{r}\frac{dx}{ds}. \end{aligned} \right\} . \quad (3)$$

If p is the perpendicular ON let fall from O on the tangent at P, and ρ is the radius of curvature of the curve at P, we have, page 88, Vol. I, Kinematics,

$$\rho = \frac{rdr}{dp}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now from equation (1), page 112, if F is the force upon the element ds, we have

$$T_2 - T_1 = dT = - F \cos \phi,$$

or, substituting the value of $\cos \phi$ from (3),

$$dT = - \frac{F}{rds}(xdx + ydy).$$

But $x^2 + y^2 = r^2$, hence $xdx + ydy = rdr$, and therefore

$$dT = - \frac{F}{ds}dr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From equation (2), page 112, we have

$$\frac{T}{\rho} = \frac{F}{ds} \sin \phi,$$

or, substituting the value of $\sin \phi$ from (3),

$$\frac{T}{\rho} = \frac{F}{rds}\left(x\frac{dy}{ds} - y\frac{dx}{ds}\right).$$

But $x\frac{dy}{ds} - y\frac{dx}{ds} = x \sin \alpha - y \cos \alpha = p$. Therefore

$$Tr = \frac{F}{ds}\rho p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Substituting the value of $\frac{F}{ds}$ from (5), we obtain

$$dT = -\frac{Trdr}{\rho p}.$$

Substituting the value of ρ from (4), we obtain

$$\frac{dT}{T} = -\frac{dp}{p}.$$

If we integrate this and let $T = T_1$ when $p = p_1$, we have

$$Tp = T_1 p_1 = \text{a Constant.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence we see *that the moment of the tension with respect to the centre of force is constant, or the tension varies inversely as the perpendicular p on the tangent from the centre of force.**

Eliminating T between (7) and (6) and putting for ρ its value from (4), we have

$$\frac{dp}{p^2} = \frac{dr}{T_1 p_1} \cdot \frac{F}{ds};$$

or integrating,

$$\frac{1}{p} = -\int \frac{dr}{T_1 p_1} \cdot \frac{F}{ds}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

the limits of the integral being given by the conditions of the problem. If the force is away from the centre, or repulsive, F is positive; if towards the centre, or attractive, F is negative.

From (8), when F is given, the equation to the curve is to be found, or, if the curve is given, F may be found.

Also from (7) and (5) the tension at any point of the curve may be found.

From equation (46), page 88, Vol. I, Kinematics, we have

$$p^2 = \frac{r^4 d\theta^2}{dr^2 + r^2 d\theta^2}, \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{dr^2}{r^4 d\theta^2};$$

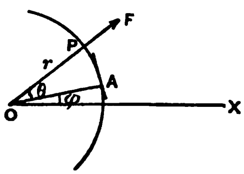
or if we denote $\frac{1}{r}$ by u,

$$\frac{1}{p^2} = u^2 + \frac{du^2}{d\theta^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Equation (9) will be found useful in reductions.

* Compare with page 85, Vol. I, Kinematics, where we see that for a particle moving with central acceleration the *moment of the velocity* is constant.

[Central Force Inversely as the Square of the Distance.]—As an application of the preceding Article, let us suppose the force F upon an element ds of the string to be repulsive and to vary inversely as the square of the distance from the centre of force.



Let δ be the density of the string or the mass of a unit of volume. Then the mass of an element of length ds whose area of normal section is A is $\delta A ds$. Let the central acceleration of one unit of mass at a known distance of r' from the centre be a'. Then the acceleration a at any distance r is given by

$$\frac{a}{a'} = \frac{r'^2}{r^2}, \quad \text{or} \quad a = \frac{a'r'^2}{r^2}.$$

The force F upon an element ds at the distance r is

$$F = \pm \frac{a'r'^2}{r^2} \cdot \delta A ds,$$

where the $(+)$ sign is to be taken for repulsive force and the $(-)$ sign for attractive force. Let the density δ and area A of normal section be constant, and let

$$\mu = a'r'^2\delta A, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where the constant μ is evidently numerically equal to the force on the mass of one unit of length of the string at a distance unity.

Then if the force is repulsive, we have

$$\frac{F}{ds} = + \frac{\mu}{r^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From equation (5), page 123,

$$dT = -\frac{\mu}{r^2}dr.$$

Integrating,

$$T = \frac{\mu}{r} + \text{Const.}$$

When the initial value of r is r_1, let the corresponding value of T be T_1. Then Const. $= T_1 - \frac{\mu}{r_1}$, and we have*

$$T = T_1 + \mu\left(\frac{1}{r} - \frac{1}{r_1}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From equation (8), page 124,

$$\frac{1}{p} = -\int \frac{\mu}{T_1 p_1} \cdot \frac{dr}{r^2} = -\int \frac{\mu}{m_1} \cdot \frac{dr}{r^2},$$

where we denote the moment $T_1 p_1$ by m_1.

Integrating,

$$\frac{1}{p} = \frac{\mu}{m_1} \cdot \frac{1}{r} + \text{Const.}$$

---

* Notice the analogy with the velocity as given on page 145, Vol. I, Kinematics, of a particle acted upon by an *attractive* force varying inversely as the square of the distance, viz.,

$$v^2 = v_1^2 + 2a'r'^2\left(\frac{1}{r} - \frac{1}{r_1}\right).$$

Let $p = p_1$ when $r = r_1$. Then Const. $= \dfrac{1}{p_1} - \dfrac{\mu}{m_1 r_1}$ and

$$\frac{1}{p} = \frac{1}{p_1} + \frac{\mu}{m_1}\left(\frac{1}{r} - \frac{1}{r_1}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

If we put for the sake of simplicity

$$\frac{\mu}{m_1} = c \quad \text{and} \quad \frac{1}{p_1} - \frac{\mu}{m_1 r_1} = -c\kappa, \quad . \quad . \quad . \quad . \qquad (5)$$

equation (4) becomes

$$\frac{1}{p} = \frac{c}{r} - c\kappa;$$

or if we denote $\dfrac{1}{r}$ by u,

$$\frac{1}{p} = cu - c\kappa. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

We have then from equation (9), page 124,

$$\frac{1}{p^2} = u^2 + \frac{du^2}{d\theta^2} = c^2(u - \kappa)^2.$$

Hence

$$\frac{du^2}{d\theta^2} = (c^2 - 1)u^2 - 2c^2\kappa u + c^2\kappa^2. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

The integral of this equation will give the equation of the curve of equilibrium.

We have evidently three cases: when $c^2 > 1$; when $c^2 = 1$; when $c^2 < 1$.

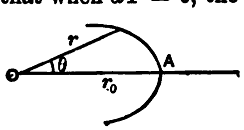
Case I: When c^2 is Greater than Unity.—Let c^2 be greater than unity. Then let

$$c^2 - 1 = n^2,$$

and we have from equation (6), after reduction,

$$d\theta = \frac{du}{n\sqrt{\left(u - \dfrac{c^2\kappa}{n^2}\right)^2 - \dfrac{c^2\kappa^2}{n^4}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

From equation (3) we have $dT = \mu d\mu$. But we have seen, page 112, that when $dT = 0$, the force is normal to the curve of the string. That value of u in equation (8) which makes $du = 0$ *will then give an apse A*, that is, a point where the string is perpendicular to the force. Let this value of u be $u_0 = \dfrac{1}{r_0}$.



From equation (8), putting $\dfrac{du}{d\theta} = 0$, we obtain

$$u_0 = \frac{1}{r_0} = \frac{c\kappa}{n^2}(1 + c). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

For any value of u *less* than this, equation (8) becomes imaginary. *All values of u must therefore be greater than u_0*, that is, u increases or r diminishes each way from the apse. We have then *du positive* in equation (8).

Integrating equation (8), we obtain

$$\theta = \frac{1}{n}\operatorname{logn}\left[u - \frac{c^2\kappa}{n^2} + \sqrt{\left(u - \frac{c^2\kappa}{n^2}\right)^2 - \frac{c^2\kappa^2}{n^4}}\right] + \text{Const.}$$

Le $\theta = \phi$ when $u = u_0 = \frac{c\kappa}{n^2}(1 + c)$. Then Const. $= \phi - \frac{1}{n}\operatorname{logn}\frac{c\kappa}{n^2}$.

If $e = 2.718282$ is the base of the Naperian system of logarithms, we have

$$\frac{c\kappa}{n^2}e^{n(\theta - \phi)} - \left(u - \frac{c^2\kappa}{n^2}\right) = \sqrt{\left(u - \frac{c^2\kappa}{n^2}\right)^2 - \frac{c^2\kappa^2}{n^4}}.$$

Squaring and reducing, we have

$$u = \frac{c\kappa}{n^2}\left(c + \frac{e^{n(\theta - \phi)} + e^{-n(\theta - \phi)}}{2}\right). \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Equation (10) is the polar equation of the curve of equilibrium.

The values of c and κ are given by equations (5) and (1).

If we measure θ from the initial radius vector r_1 through the apse, we have $\phi = 0$, and $u_1 = u_0$. Therefore, from (9),

$$u_1 = \frac{c\kappa}{n^2}(1 + c), \quad \text{or} \quad \kappa = \frac{n^2u^1}{c(1 + c)}.$$

Substituting this value of κ in (10), we obtain

$$u = \frac{u_1}{\frac{\mu}{m_1} + 1}\left[\frac{\mu}{m_1} + \frac{e^{n\theta} + e^{-n\theta}}{2}\right]. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Equation (11) is the polar equation of the curve of equilibrium when the angle θ is measured from the initial radius vector r_1 through the apse. We have $u = \frac{1}{r}$, $u_1 = \frac{1}{r_1}$, and the value of μ is given by equation (1).

Case 2: When c^2 is Equal to Unity. — When $c^2 = 1$, we have $c = +1$ or $c = -1$. When $c = +1$, we have $n^2 = c^2 - 1 = 0$, and from equation (11), $u = u_1$, or $r = r_1$. *The centre of equilibrium when* $c = +1$ *is therefore a circle.*

When $c = -1$, we have also $n^2 = c^2 - 1 = 0$, and, from equation (11), $u = \frac{0}{0}$ or indeterminate. In this case we have, from equation (7),

$$d\theta = \frac{-du}{\kappa\sqrt{1 - \frac{2u}{\kappa}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Putting $\frac{du}{d\theta} = 0$, we have for the value of u at the apse

$$u_0 = \frac{\kappa}{2}.$$

For any value of u greater than this equation (12) is imaginary. All values of u must then be less than u_0, or u diminishes each way from the apse. Hence du is negative.

Integrating (12), we have

$$\theta = \sqrt{1 - \frac{2u}{\kappa}} + \text{Const.}$$

Let $\theta = \phi$ when $u = u_0 = \frac{\kappa}{2}$. Then Const. $= \phi$, and

$$\theta - \phi = \sqrt{1 - \frac{2u}{\kappa}}, \quad \text{or} \quad u = \frac{\kappa}{2}[1 - (\theta - \phi)^2].$$

Hence

$$r = \frac{\frac{2}{\kappa}}{1 - (\theta - \phi)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Equation (13) is the polar equation of the curve of equilibrium when $c = -1$. The value of κ is given by (5).

If we measure θ from the initial radius vector r_1 through the apse, we have $\phi = 0$, and $u_1 = \frac{\kappa}{2} = \frac{1}{r_1}$, or $\kappa = \frac{2}{r_1}$. Substituting this value of κ, we have

$$r = \frac{r_1}{1 - \theta^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Equation (14) is the polar equation of the curve of equilibrium when $c = -1$, when the angle θ is measured from the initial value of r through the apse.

Case 3: When c^2 is Less than Unity.—Let $c^2 < 1$ and put $1 - c^2 = n^2$. Then from equation (7), after reduction, we have

$$d\theta = \frac{-du}{u\sqrt{\frac{c^2\kappa^2}{n^4} - \left(u + \frac{c^2\kappa}{n^2}\right)^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Putting $\frac{du}{d\theta} = 0$, we have for the value of u at the apse

$$u_0 = \frac{c\kappa}{n^2}(1 - c).$$

Any value of u greater than u_0 gives equation (15) imaginary. All values of u must then be less than u_0, or u diminishes each way from the apse. Hence we take du negative in equation (15). Integrating, we have

$$\theta = \frac{1}{n}\cos^{-1}\frac{u + \frac{c^2\kappa}{n^2}}{\frac{c\kappa}{n^2}} + \text{Const.}$$

Let $\theta = \phi$ when $u = u_0$. Then Const. $= \phi$, and

$$(\theta - \phi) = \frac{1}{n}\cos^{-1}\left(c + \frac{n^2u}{c\kappa}\right),$$

or

$$\cos n(\theta - \phi) = c + \frac{n^2u}{c\kappa}.$$

Hence

$$u = -\frac{c\kappa}{n^2}[c - \cos n(\theta - \phi)]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Equation (16) is the polar equation of the curve of equilibrium.

The value of κ is given by (5).

If we measure θ from the initial radius vector r_1 through the apse, we have $\phi = 0$, and $u_1 = u_0 = \frac{c\kappa}{n^2}(1 - c)$, or $\kappa = \frac{u_1 n^2}{c(1-c)}$.

Substituting this value of κ in equation (16), we have, if we put $c = -\frac{\mu}{m_1}$, where μ is given by equation (1),

$$u = \frac{u_1}{1 + \frac{\mu}{m_1}}\left(\frac{\mu}{m_1} + \cos n\theta\right). \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Equation (17) differs from the focal polar equation of a conic only in having the angle θ multiplied by a number n less than unity.

EXAMPLES.

(1) *An endless flexible string of uniform linear density but without weight is moving so that the velocity of each element has a constant magnitude v and a direction always tangential to the string. Show that the tension is the same at every point of the string, and find it.*

Ans. Since the tangential velocity is constant, there is no tangential acceleration and hence no tangential force.

Therefore from equation (1), page 112, $T_2 - T_1 = 0$, or there is no variation in tension.

If ρ is the radius of curvature at any point, then the normal acceleration of hat point is $f_n = \frac{v^2}{\rho}$ (page 58, Vol. I, Kinematics).

If δ is the linear density, or the mass per unit of length, then the normal force per unit of length is $\delta f_n = \frac{\delta v^2}{\rho}$. From equation (2), page 112, we have then

$$\frac{\delta v^2}{\rho} = \frac{T}{\rho}, \quad \text{or} \quad T = \delta v^2,$$

where T is given in poundals. In gravitation units (page 6),

$$T = \frac{\delta v^2}{g},$$

where g is the acceleration of gravity.

(2) *An endless flexible circular string of radius r and of uniform linear density δ, but without weight, rotates in its own plane about its centre with the angular velocity ω. Find its tension.*

Ans. The tangential velocity $r\omega$ is constant, and hence there is no tangential force. Therefore, just as in the preceding example, there is no variation in tension.

The normal acceleration is $f_n = r\omega^2$ (page 76, Vol. I, Kinematics).

If δ is the mass per unit of length, then the normal force per unit of length is $\delta r\omega^2$. From equation (2), page 112, we have then

$$\delta r\omega^2 = \frac{T}{r}, \quad \text{or} \quad T = \delta r^2\omega^2,$$

where T is given in poundals. In gravitation units (page 6),

$$T = \frac{\delta r^2 \omega^2}{g},$$

where g is the acceleration of gravity.

(3) *A body weighing 7 lbs. is suspended from a fixed point by a uniform string, 12 inches long, weighing 18 oz. Find the stress in the string at its middle point and at its upper and lower ends.*

Ans. $7\frac{9}{16}$ lbs., $8\frac{1}{8}$ lbs., 7 lbs., in gravitation units; or, taking $g = 32$, 242 poundals, 260 poundals, 224 poundals.

(4) *Show that the horizontal component of the tension at any point of a uniform flexible string hanging in equilibrium from two fixed points is equal to the tension at the lowest point, and that the vertical component is equal to the weight of the portion of the string between the given point and the lowest point.*

Ans. See page 111.

(5) *Show that at any point of a uniform flexible string which is hanging in equilibrium with two points fixed, its inclination to the horizon is the angle whose tangent is the ratio of the weight of the portion of the string between the given point and the lowest point to the tension at the lowest point.*

Ans. See page 116.

(6) *In the preceding example, show that the square of the tension at any point is equal to the sum of the squares of the weight of the portion of the string between the given point and the lowest point, and of the tension at the lowest point.*

Ans. See page 117.

(7) *A telegraph wire, weighing 400 lbs. per mile, is stretched between two points in the same horizontal line at a distance of 100 yds. with a horizontal tension of 400 lbs. Find the deflection of the lowest point of the wire below the fixed points, neglecting stretch and supposing the wire perfectly flexible.*

Ans. From equation (6), page 116, $x = 150$ ft., $c = 5280$ ft., deflection $=$ 2.1 ft.

(8) *A uniform wire weighs w lbs. per foot and is just able to stand a stress of P pounds. It is hung between two points in the same horizontal line, distant d ft., so as to be on the point of breaking. Obtain an equation to determine the half length s, the wire being supposed to be perfectly flexible and inextensible.*

Ans. From page 117 we have $P^2 = H^2 + w^2s^2$. Hence $H = \sqrt{P^2 - w^2s^2}$. Also $c = \frac{H}{w} = \frac{\sqrt{P^2 - w^2s^2}}{w}$, and $x = \frac{d}{2}$. Therefore, from equation (8),

$$s = \frac{\sqrt{P^2 - w^2s^2}}{2w}\left(e^{\frac{dw}{2\sqrt{P^2 - w^2s^2}}} - e^{-\frac{dw}{\sqrt{2P^2 - w^2s^2}}}\right).$$

(9) *A string 202 ft. long, which weighs 1 lb. for every 10 ft., is hung between two points in the same horizontal line distant 200 ft. Obtain an equation to determine the tension H at the lowest point in gravitation units.*

Ans. We have $s = 101$ ft., $w = \frac{1}{10}$ lb., $x = 100$ ft., $c = \frac{H}{w} = 10H$.

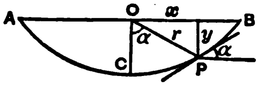
From equation (8), page 117,

$$101 = 5H\left(e^{\frac{10}{H}} - e^{-\frac{10}{H}}\right).$$

Solving this equation by a series of approximations, we find H to be about 40 lbs., provided the string is perfectly flexible and inextensible.

(10) *Find the law of variation of the mass per unit of length at each point of a string acted on by gravity in order that it may hang in the form of a semi-circle whose diameter is horizontal.*

Ans. Let $AB = 2r$ be the horizontal diameter and O the centre of the semicircle. Let P be any point of the curve, and the angle $POC = \alpha$. Let the co-ordinates of P be x and y.



Then $\cos\alpha = \dfrac{dx}{ds}$, $\sin\alpha = \dfrac{dy}{ds}$.

We have from equations (4), page 111,

$$-H + T\frac{dx}{ds} = 0,$$

$$-\int_0^s \delta ds + T\frac{dy}{ds} = 0,$$

where δ is the linear density or mass per unit of length, and H and T are in gravitation units.

Dividing the second by the first, we have

$$H\frac{dy}{dx} = \int_0^s \delta ds,$$

or

$$H\frac{d^2y}{dx^2} = \frac{\delta ds}{dx}.$$

But the equation of the curve is $x^2 + y^2 = r^2$. Hence

$$\frac{dy}{dx} = -\frac{x}{y}, \quad \text{and} \quad \frac{d^2y}{dx^2} = -\frac{y - x\dfrac{dy}{dx}}{y^2} = -\frac{r^2}{y^3}.$$

Therefore

$$\frac{\delta ds}{dx} = -H\frac{r^2}{y^3}, \quad \text{or} \quad \delta = -H\frac{r^2}{y^3}\frac{dx}{ds}.$$

But $\dfrac{dx}{ds} = \cos\alpha$, and $r\dfrac{dx}{ds} = -y$. Hence

$$\delta = +\frac{Hr}{y^2}.$$

That is, the mass per unit of length varies inversely as the square of the distance of the point below the horizontal diameter.

(11) *A telegraph line is constructed of wire which weighs 7.3 lbs. per 100 feet. The distance between the posts is 150 feet and the wire sags 1 foot in the middle. Show that it is screwed up to a tension of about 820 lbs.*

(12) *Find the law of variation of the mass per unit of length in order that a string may hang under the action of gravity in a parabola.*

Ans. From page 113, the load per unit of horizontal projection is constant and equal to w. The load per unit of length is then proportional to the tangent of the slope, or $\dfrac{F}{ds} = w \tan \alpha$.

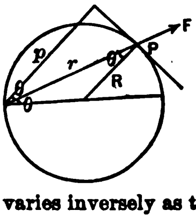
But $\tan \alpha$ is proportional to the horizontal projection of the length. Hence the mass per unit of length is proportional to the horizontal projection of the unit of length.

(13) ***Show that the area of normal section at any point in the catenary of uniform strength is proportional to the radius of curvature.***

Ans. From page 121, we see that A is proportional to $\sec \dfrac{x}{c}$. From page 120, equation (7), we see that $\sec \dfrac{x}{c}$ is proportional to the radius of curvature.

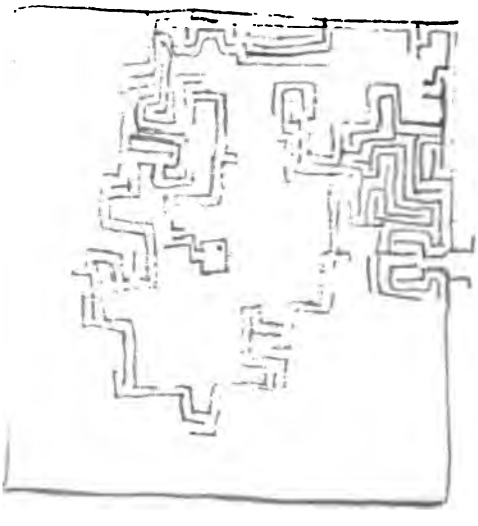
(14) ***A uniform inextensible string assumes the form of a circle under the action of a repulsive force emanating from a point on its circumference. Find the law of force.***

Ans. From page 124, $Tp = \text{Const.} = c$, or $T = \dfrac{c}{p}$. But if r is the radius



vector of any point P, $p = r \cos \theta$. Hence $T = \dfrac{c}{r \cos \theta}$.

From page 112, $\dfrac{T}{R} = \dfrac{F}{ds} \cos \theta$, where R is the radius of the circle. Hence $\dfrac{F}{ds} \cdot R \cos \theta = \dfrac{c}{r \cos \theta}$. But $R \cos \theta = \dfrac{1}{2} r$ and $\cos \theta = \dfrac{1}{2} \dfrac{r}{R}$. Hence $\dfrac{F}{ds} = \dfrac{4cR}{r^3}$, or the force varies inversely as the cube of the distance.



CHAPTER VI.

GRAPHICAL STATICS—CO-PLANAR FORCES.

CONCURRING CO-PLANAR FORCES. APPLICATION TO FRAMED STRUCTURES. APPARENT INDETERMINATION. NON-CONCURRING FORCES. EQUILIBRIUM POLYGON. GRAPHIC CONSTRUCTION FOR CENTRE OF PARALLEL FORCES. PROPERTIES OF EQUILIBRIUM POLYGON. APPLICATION TO PARALLEL FORCES.

Graphical Statics.—While the solution of statical problems by computation and analytical methods is sometimes tedious and involved, they may often be solved with comparative ease and sufficient accuracy by graphic construction.

The solution of statical problems by graphic methods gives rise to **graphical statics.** We shall consider only co-planar forces.

Concurring Co-planar Forces. — Let any number of co-planar forces F_1, F_2, F_3, F_4, etc., given in magnitude and direction, act at a point A, Fig. 1.

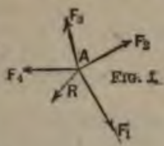


FIG. 1.

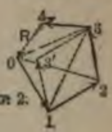


FIG. 2.

In Fig. 2, from any point 0, lay off to scale the line representative of F_1 from 0 to 1, then the line representative of F_2 from 1 to 2, then the line representative of F_3 from 2 to 3, then the line representative of F_4 from 3 to 4, and so on. The polygon 0 1 2 3 4 thus obtained we call the **force polygon.**

If all these forces are in equilibrium, the algebraic sum of their horizontal and vertical components must be zero. But when this is the case, evidently 4 and 0, in Fig. 2, must coincide, or the *force polygon must close.* We have then the following principle:

If any number of concurring forces are in equilibrium, the force polygon is closed. If the force polygon is not closed, the line 0 4 *necessary to make it close gives the magnitude and direction of the resultant R. If we consider this resultant acting at the point of application A in the direction from* 4 *to* 0, *obtained by following round the polygon in the direction of the forces, it will hold the forces at A in equilibrium. If taken as acting in the opposite direction at A, it will replace the forces.*

COR. 1. The order in which the forces are laid off in the force polygon is immaterial. Thus in Fig. 2, if we had laid off 0 1, then the line representative of F_3 from 1 to 3′, and then the line representative of F_2, we should arrive at 3 just as before. By a similar change of two and two we can have any order we please.

COR. 2. Any line in the force polygon, as 0 2, 0 3, or 1 3, is the resultant of the forces on either side. Thus 0 2 is the resultant of F_1 and F_2, and, acting in the direction from 2 to 0, holds F_1 and F_2 in equilibrium and replaces F_3, F_4 and R.

COR. 3. If the forces are all parallel, the force polygon becomes a straight line. Thus in Fig. 1, if the parallel forces F_1, F_2, F_3, F_4, etc., act at the point A, we have the force polygon Fig. 2, 0 1 2 3 4, and the closing line 4 0 is as before, the resultant R and equal to the algebraic sum of the forces.

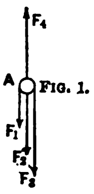


FIG. 1.

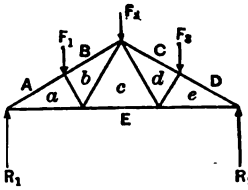
FIG. 2.

If taken as acting from 4 to 0, it will hold the forces at A in equilibrium. In the opposite direction it will replace the forces.

Notation for Framed Structures. — Let the figure represent a roof-truss composed of two rafters, a horizontal tie-rod and intermediate braces consisting of struts and ties.

The notation which we adopt in order to designate any number of a framed structure, or any force acting upon the structure, is as follows:

We place a letter in each of the triangular spaces into which the frame is divided by the members, and also a letter between any two forces. Any number or force is then denoted by the letter *on each side of it.* Thus in the figure AB denotes the force F_1, BC denotes the force F_2, CD denotes the force F_3, DE denotes the upward pressure of the right-hand support R_2, EA denotes the upward pressure of the left-hand support R_1. Also Aa, Bb, Cd, De denote the portions of the rafters which have these letters on each side. The portions into which the lower tie is divided are in the same way Ea, Ec, Ee. The braces are ab, bc, cd, de.



The student should carefully adhere to this notation for the frame *whenever using the graphic method.*

Character of the Stresses.—The determination of the *kind* of stress in a member of a frame, whether tension or compression, is as important as the determination of the magnitude of the stress.

In the preceding figure, suppose we know the upward pressure at the left support R_1 or EA, and we wish to find the stresses in the members Ea and Aa, Fig. 1, which meet at the lower left-hand

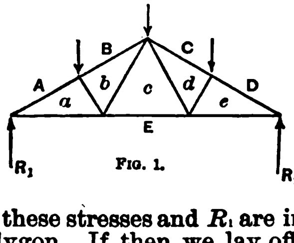


FIG. 1.

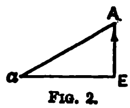


FIG. 2.

apex. If these stresses and R_1 are in equilibrium, they will make a closed polygon. If then we lay off EA in Fig. 2, upwards, equal to R_1, and then from A and E draw lines parallel to Aa and Ea in Fig. 1, and produce them till they intersect at a, Fig. 2, evidently the lines Aa and Ea in Fig. 2, taken to the same scale as EA, will give the magnitude of the stresses in Ea and Aa in Fig. 1.

Thus, *lines in the force polygon which have letters at each end give the stresses in those members of the frame denoted by the same letters at the sides.*

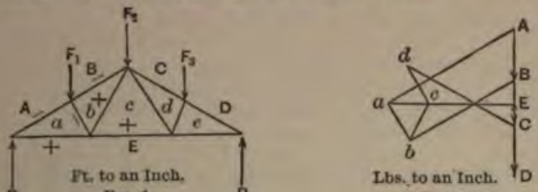
Now as to the *character* of these stresses, the directions Aa and aE in Fig. 2, obtained by following round in the known direction of R_1, are the directions for equilibrium (page 133).

Since we are considering the concurring forces acting at the left-hand apex, transfer these directions to Fig. 1, and we see that Aa acts towards the apex we are considering and thus resists compression, and aE acts away from it and therefore resists tension. The stress in Aa is therefore compressive $(-)$ and in aE tensile $(+)$.

In general, then, if we take any apex of the frame in Fig. 1, and consider the concurring forces acting at that apex as a system of concurring forces in equilibrium, we have the following rule:

Follow round the force polygon in Fig. 2 in the direction indicated by any one of these forces already known, and transfer the directions thus obtained for the stresses to the apex in Fig. 1 under consideration. If the stress in any member is thus found acting away from the apex, it is tension $(+)$; *if towards the apex, it is compression* $(-)$.

Application of Preceding Principles to a Frame. — Let Fig. 1 be a frame consisting of two rafters, a horizontal tie-rod and bracing as shown, carefully drawn to a scale of a certain number of feet to an inch. This we call the **frame diagram.**



Ft. to an Inch.
FIG. 1.

Lbs. to an Inch.
FIG. 2.

Let the forces F_1, F_2, F_3 act at the upper apices, and let the reactions or upward pressures of the supports be R_1 and R_2. Notate the frame and these forces as directed, so that $F_1 = AB$, $F_2 = BC$, $F_3 = CD$, $R_2 = DE$, $R_1 = EA$, while the members are Aa, Bb, Cd, De, Ee, Ec, Ea, ab, bc, cd, de.

The outer forces acting upon the frame cause stresses in the members. These outer forces must first be all known, or if any are unknown, they must first be found.

Lay off these outer forces AB, BC, CD, DE, EA in Fig. 2 to a scale of a certain number of pounds to an inch. Each force in Fig. 2, having letters *at its ends, is equal and parallel to those forces in Fig.* 1 *which have the same letters at the sides.*

The polygon formed by AB, BC, CD, DE, EA (in this case a straight line, Cor. 3, page 134) we have called the **force polygon.**

If the frame is in equilibrium, this polygon *must always close*, that is, the outer forces acting upon the frame must be in equilibrium. If it does not close, these outer forces are not in equilibrium and the frame will move. That is, the frame itself, so far as its motion as a whole is considered, may be treated as a point (page 83).

Having thus drawn and notated the frame Fig. 1 and constructed the force polygon Fig. 2, we can find the stresses in the members. The forces and stresses *at each apex* must be in equilibrium, and therefore form a closed polygon.

Thus consider first the left-hand apex, Fig. 1. At this point we have the reaction EA and the stresses in Aa and Ea, constituting a system of concurring forces in equilibrium. But we already have EA laid off in Fig. 2. If then we draw Aa and Ea in Fig. 2 parallel to Aa and Ea in Fig. 1, and produce to intersection a, the polygon is closed and we have in Fig. 2 the stresses in Aa and Ea, to the same scale employed in laying off EA. Since EA acts upwards, if we follow round from E to A and A to a, and a to E, in Fig. 2, and transfer the directions thus obtained for Aa and aE to the left-hand apex in Fig. 1, we have the stress in Aa towards this apex or compression $(-)$, and the stress in aE away from the apex and therefore tension $(+)$.

[The student should follow with his own sketch and mark each stress with its proper sign as he finds it.]

Let us now pass to the next upper apex, at F_1, Fig. 1. Here we have F_1 or AB and the stresses in Aa, ab and Bb in equilibrium. But we already have the stresses in Aa and AB laid off in Fig. 2.

If then we draw from a and B in Fig. 2 lines parallel to ab and Bb in Fig. 1, and produce to intersection b, the polygon is closed and we have in Fig. 2 the stresses in ab and Bb. Since AB is known to act downward, we follow round in Fig. 2, from A to B, B to d, d to a, and a to A, and transfer the directions thus obtained to the apex at F_1, Fig. 1, under consideration. We thus obtain the stress in Bb towards the apex or compression, the stress in ba towards the apex or compression, and the stress in aA towards the apex or compression, *just as already found.*

Note that in the first case, when we were considering the apex at R_1, we found the stress in aA acting towards that apex. Now when we consider the apex at F_1 we find the stress in aA acting towards that apex—in both cases, then, compression.

Let us now consider the second lower apex, Fig. 1. We have here no outer force, but the stresses in Ea, ab, bc and cE must be in equilibrium and therefore form a closed polygon. But in Fig. 2 we have already found the stresses in Ea and ab. If then we draw from b a line parallel to bc in Fig. 1, and produce it to intersection c with Ea, the polygon closes, and we have in Fig. 2 the stresses in bc and cE. We have already found aE to be tension. It must therefore act away from the apex we are considering. We therefore follow round in Fig. 2, from E to a, a to b, b to c, and c to E, and transfer the directions thus found to the corresponding members in Fig. 1. We thus obtain the stress in Ea tension and the stress in ab compression as already found, and the stress in bc tension and in cE tension.

Let us now consider the top apex. We have here the force $F_2 = BC$, and the stresses in Bb, bc, cd and dC, in equilibrium. But in Fig. 2 we have already laid off BC, and we have found the stresses in Bb and bc. If then we draw from c and C lines parallel to cd and Cd in Fig. 1, and produce to intersection d, the polygon closes and we have in Fig. 2 the stresses in cd and Cd. Since BC acts downwards, we follow round from B to C, C to d, d to c, c to b, and b to B. Transferring these directions to the corresponding members in Fig. 1, we obtain the stress in Cd compression and in dc tension, while the stress in cb is tension and in bB compression as already found.

We can thus go to each apex and find the stresses in every member.

The lines in Fig. 2 which thus give the stresses in the members constitute the **stress diagram**. *Each stress having letters at its*

ends in Fig. 2 is parallel to that member in Fig. 1 *which has the same letters at its sides.*

Apparent Indetermination of Stresses.—It sometimes happens that a frame has no superfluous members and yet in applying the graphic method we are unable to find any apex at which all the forces but two are known. In such case the difficulty may be overcome by taking out one or more of the members and replacing them by another member, and then applying the method until we find the stress in some member which *is not affected by the change.* Or we may find the stress in this member by the method of sections (page 102). Having found this stress, we can replace the members taken out and find the actual stresses.

Thus let Fig. 1 be a frame* acted upon by the forces F_1, F_2, F_3, F_4, etc., and the reactions or upward pressures of the supports R_1, R_2.

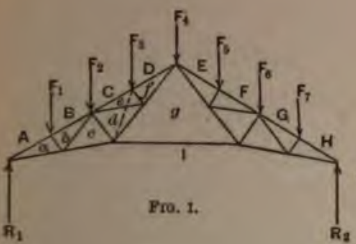


Fig. 1.

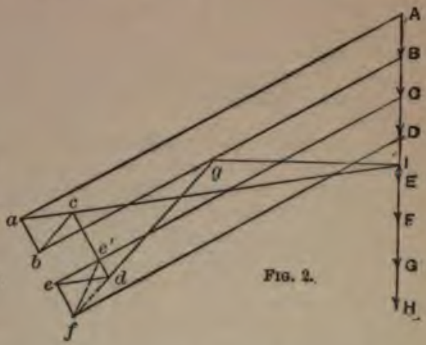


Fig. 2.

Notate the frame and the forces by letters on each side as directed (page 134).

Then lay off to scale the outer forces in Fig. 2, thus forming the force polygon *ABCD . . . HIA*. This polygon is a straight line in this case, because all the forces are parallel, and it must close, that is, the outer forces are in equilibrium.

We can now proceed to find the stresses as follows:

Consider first the left-hand apex, Fig. 1. At this point we have the reaction *IA* and the stresses in *Aa* and *Ia* constituting a system of concurring forces in equilibrium. But we already have *IA* laid off in Fig. 2. If then we draw *Aa* and *Ia* in Fig. 2 parallel to *Ia* and *Aa* in Fig. 1, and produce to intersection *a*, the polygon is closed and we have in Fig. 2 the stresses in *Aa* and *Ia* to the same scale employed in laying off the forces. Since *IA* acts upwards, we follow round from *I* to *A*, *A* to *a*, and *a* to *I*, in Fig. 2, and transfer the directions thus obtained for *Aa* and *aI* to the corresponding members in Fig. 1.

We have then the stress in *Aa* towards the apex we are considering or compression (−), and the stress in *aI* away from that apex or tension (+).

Considering now the next upper apex, we have here the force *AB* known, the stress in *Aa* already found, and the stresses in *ab* and *Bb* unknown. If then in Fig. 2 we draw *ab* and *Bb*, thus closing the polygon, we obtain the stresses in *ab* and *Bb*.

---

* Disregard for the present the dotted member in Fig. 1.

Since AB acts down, we follow round in Fig. 2 from A to B, B to b, b to a, and a back to A, and transfer the directions thus obtained to the corresponding members in Fig. 1. We have then the stress in Bb towards the apex we are considering or compression $(-)$, the stress in ba towards that apex or compression $(-)$, and the stress in aA also towards that apex or compression $(-)$, just as we have already found it.

Note that when we were considering the apex at R_1, we found the stress in aA acting towards that apex. Now when we consider the apex at F_1 we find the stress in aA acting towards that apex. In both cases, then, compression.

We can now consider the next lower apex, where we have the stresses in Ia, ab, bc and cI in equilibrium. We already know Ia and ab, and if we draw in Fig. 2 bc and cI, we obtain the stresses, in bc tension $(+)$, and in cI tension.

Thus far there has been no difficulty in the application of the graphic method. But now we cannot consider the next upper or lower apex, because at each we have more than two unknown forces. If we should start at the right end, we should soon come to the same difficulty on the right side. Apparently we can go no farther.

The number of members is 27 (we disregard the dotted member in Fig. 1). The number of apices is 15. We have then, applying the criterion for superfluous members (page 103), $m = 2n - 3$. There are then no superfluous members.

If now *we remove* the two members de and ef and replace them by the dotted member $e'f$, where e' takes the place in the new notation of the two letters e and d, we have still a rigid frame with no superfluous members. For the number of members is now $m = 25$ and the number of apices is $n = 14$. We have then $m = 2n - 3$.

But this change has evidently *not affected the stress in the member* Ig. We can therefore now carry on the diagram until we find the stress in Ig, or we may compute the stress in Ig directly by the method of sections (page 102).

Thus if we now consider the apex at F_2, Fig. 1, we have at this point the stresses in the members Bb, bc, ce' and $e'C$, and the force BC, all in equilibrium. We know BC, Bb and bc, and if we draw in Fig. 2 ce' and $e'C$, we obtain the stresses in $e'C$ compression and in ce' compression.

We can then pass to the apex at F_3, Fig. 1, where we know all the forces except the stresses in Df and fe'. We draw then Df and fe' in Fig. 2, and obtain the stresses in Df compression and in fe' tension.

We can now pass to the next lower apex, where we have the stresses in Ic, ce' and $e'f$, and can therefore find fg and Ig. We draw then fg and Ig in Fig. 2, and obtain the stresses in fg and Ig tension.

We have thus found the stress in the member Ig, and since this is unchanged by the removal of the members de and ef, we can now replace those members and remove $e'f$.

We can now consider the second lower apex and find the stresses in cd and dg, and can then pass to the apex at F_3 and find the stresses in ef and Df, and so on. We can thus find the stress in every member of the frame, and there is no real indeterminateness.

Remarks upon the Method.—The method just illustrated we may call the "*graphic method by resolution of forces.*" The student will note that he must always know all but two of the forces concurring at any apex before he can consider that apex.

It is evident that if the frame is completely divided into two portions by cutting the members, the stresses which existed in the cut members before the section was made must hold in equilibrium the outer forces acting upon each portion of the frame (page 102).

This is at once made evident by Fig. 2, page 137.

Thus suppose a section cutting the members Bb, bc and cE, Fig. 1, and thus dividing the frame into two portions. We see from Fig. 2 that the stresses in the cut pieces make a closed polygon with EA and AB, the outer forces on the left-hand portion, or with BC, CD and DE, the outer forces on the right-hand portion.

If we solve the triangles in Fig. 2, page 137, we obtain algebraic expressions for the stresses identical with those obtained by the "algebraic method by resolution of forces" (page 101).

Thus since the algebraic sum of the horizontal and vertical components of the forces acting at each apex must be zero, we have $+ R_1 + Aa \cos \alpha = 0$, or $Aa = - \dfrac{R_1}{\cos \alpha}$, where α is the angle of the rafter with the vertical. We get the same result at once from Fig. 2 by solving the triangle AaE. In the same way we have at once, from Fig. 2, $ab = - F_1 \cos \beta$, where β is the angle of ab with the vertical.

We see also from Fig. 2, page 137, other relations. Thus we see that the stress in ab will be the least possible when it is perpendicular to the rafter. We also see at a glance how the stress in any member is affected by a change of inclination of the member.

Finally, the application of the method is equally simple no matter how irregular the frame may be.

If the frame is symmetrical with respect to the centre, and the forces F_1, F_2 in Fig. 2 (page 137) are equal, it is evident that the stresses in each half will be the same. We have then $Cd = Bb$, $cd = cb$, and so on.

Choice of Scales, etc.—In general the larger the frame is drawn in Fig. 1, the better, as it then gives more accurately the direction of the members composing it.

The force polygon Fig. 2, on the other hand, should be taken to no larger scale than consistent with scaling off the forces to the degree of accuracy required, so as to avoid the intersection of very long lines, where a slight deviation from true direction multiplies the error. If an error of one twenty-fifth of an inch is considered the allowable limit, the scale should be so chosen that one twenty-fifth of an inch shall represent a small number of pounds, within the degree of accuracy required.

The stress polygon Fig. 2 should be completely finished and the signs for tension (+) and compression (—) placed on *the frame* for each member *as its stress is found*, to avoid confusion, before the stresses are taken off to scale. A good scale, dividers, straight-edge, triangle, and hard fine-pointed pencil are all the tools required. The work should be done with care, all lines drawn light, points of intersection accurately located and the frame properly notated to correspond with the force polygon. Care should be exercised to secure perfect parallelism in the lines of the frame and stress polygon. Some practice is necessary in order to obtain close results. It should be remembered that careful habits of manipulation, while they tend to give constantly-increased skill and more accurate results, affect very slightly the rapidity and ease with which these results are obtained.

EXAMPLES.

(1) *A roof-truss has a span of 50 feet and rise of 12.5 feet. Each rafter is divided into four equal panels, and the lower horizontal tie into six equal panels. The bracing is as shown in the figure. A weight of 800 lbs. is sustained at each upper apex. Find the stresses.*

Ans. Draw the frame in Fig. 1 to a scale of, say, 12 feet to an inch, and notate it. Then construct the force polygon *ABCDEFGHIA*, Fig. 2.

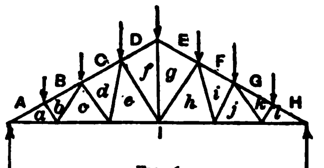


FIG. 1.

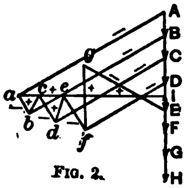


FIG. 2.

Note that R_2 or *HI* and R_1 or *IA* are equal and each 2800 lbs. The force polygon then closes as it should. We can take the scale of Fig. 2 as 3200 lbs. to an inch. Then an error of $\frac{1}{25}$ of an inch will be about 128 lbs.

We can then find the stresses as shown in Fig. 2.

Aa	*Bb*	*Cd*	*Df*	*Ia*
− 6280	− 5816	− 4700	− 3580	+ 5624
Ic	*Ie*	*ab*	*bc*	*cd*
+ 4832	+ 4024	− 720	+ 720	− 1060
de	*ef*	*fg*		
+ 928	− 1452	+ 2400 lbs		

The accurate results (Ex. (3), page 542) as found by computation are

− 6260	− 5813	− 4696	− 3577	+ 5600
+ 4802	+ 4003	− 720	+ 720	− 1081
+ 920	− 1443	+ 2401 lbs.		

It will be seen that the greatest error is only 30 lbs. The above results were actually obtained from the diagram, using the scales given.

(2) *Sketch the stress diagram for a roof-truss as shown in the following Fig.* 1, *equal forces acting at every upper and lower apex.*

Ans. The student should note that the reactions *DE* and *GA* are each equal to half the sum of the downward forces or $2\frac{1}{2}$ forces.

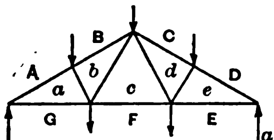


FIG. 1.

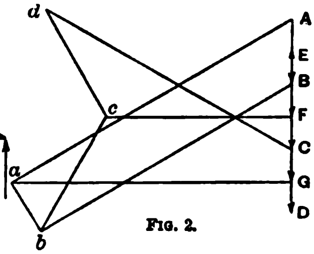


FIG. 2.

We lay off then in Fig. 2 *AB*, *BC*, *CD* downwards. Then *DE* upwards equal to $2\frac{1}{2}$ forces. Then *EF*, *FG* downwards. Then *GA* upwards equal to $2\frac{1}{2}$ forces, and *closing the force polygon*.

The stresses can now be found as always.

(3) *We give in the following figures a number of frames with their stress diagrams.* For the sake of generality, the outer forces and reactions are often taken inclined as well as vertical.*

* The student should sketch the stress diagrams for himself in each case, putting down as he goes along the sign (−) and (+) for compression and tension upon each member of the frame as soon as he finds it.

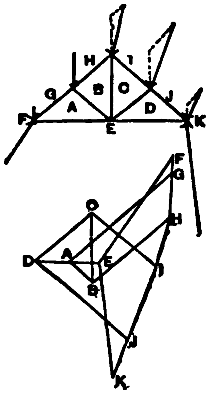


Fig. 1.

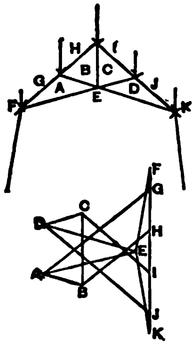


Fig. 2.

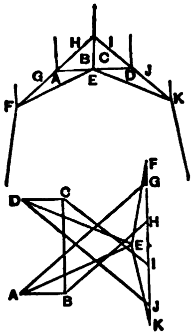


Fig. 3.

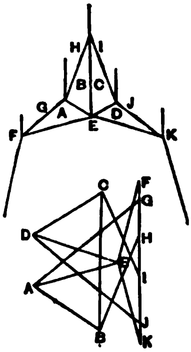


Fig. 4.

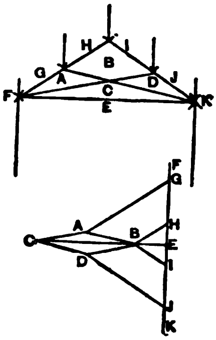


Fig. 5.

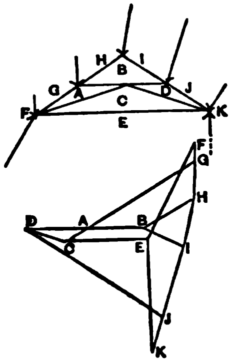


Fig. 6.

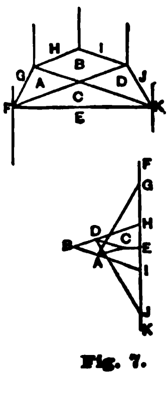


Fig. 7.

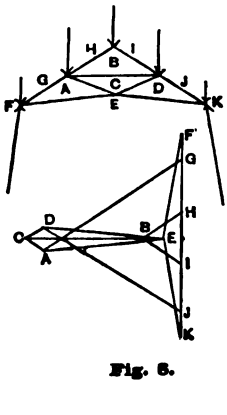


Fig. 8.

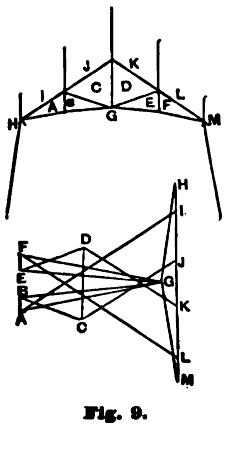


Fig. 9.

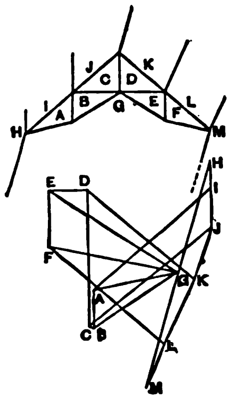


Fig. 10.

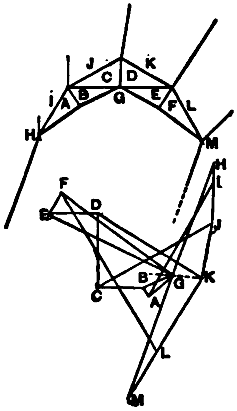


Fig. 11.

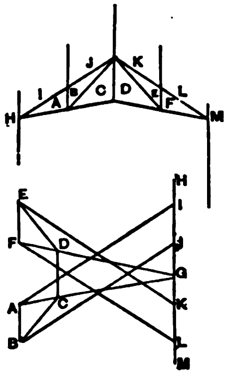


Fig. 12.

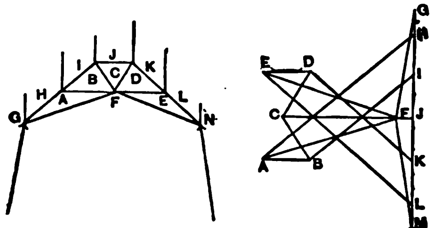


Fig. 13.

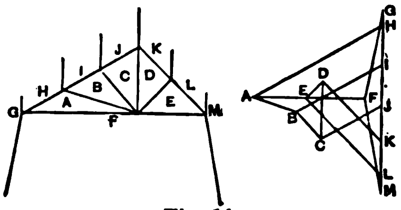


Fig. 14.

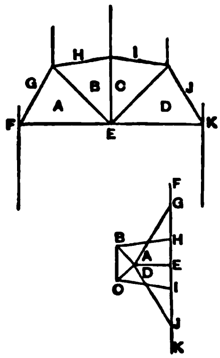


Fig. 15.

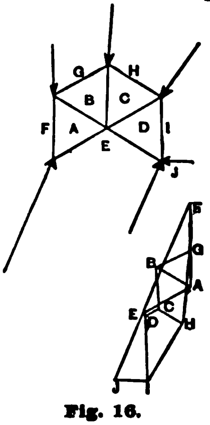


Fig. 16.

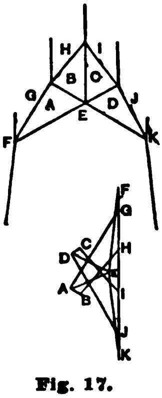


Fig. 17.

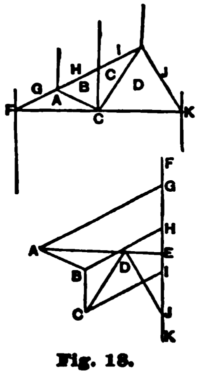


Fig. 18.

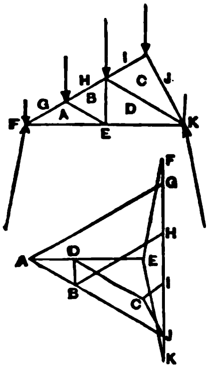


Fig. 19.

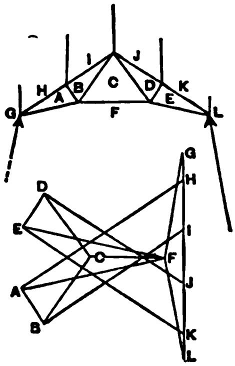


Fig. 20.

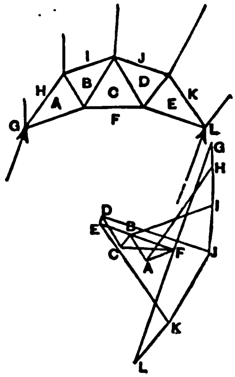


Fig. 21.

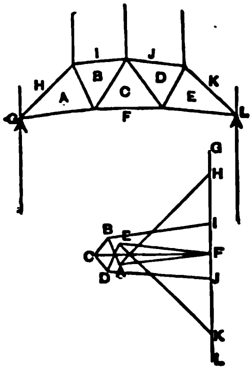


Fig. 22.

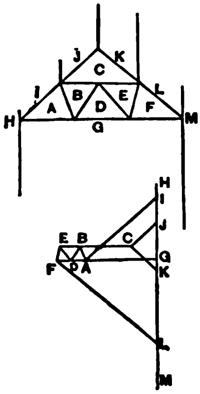


Fig. 23.

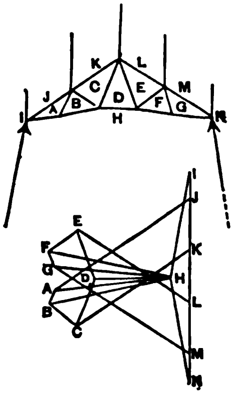


Fig. 24.

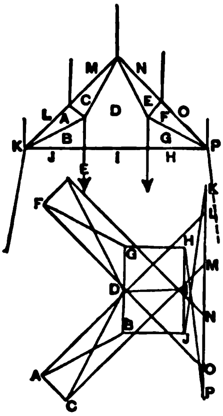


Fig. 25.

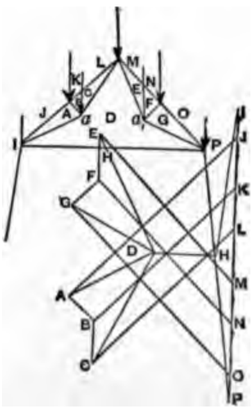


Fig. 26.

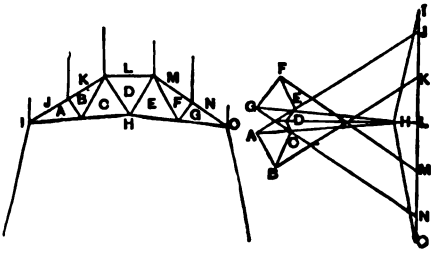


Fig. 27.

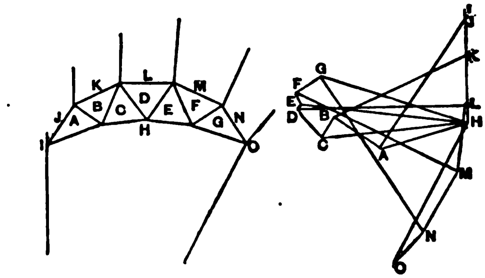


Fig. 28.

Non-concurring Forces.—Let the co-planar forces F_1, F_2, F_3, F_4, etc., act at the points A_1, A_2, A_3, A_4 of any rigid body, Fig. 1.

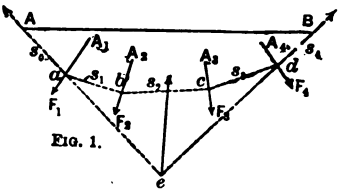


Fig. 1.

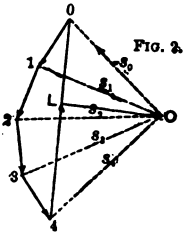


Fig. 2.

If we lay off the forces to scale in Fig. 2, we have as before the force polygon 0 1 2 3 4, and the closing line 0 4 gives as before the resultant. If this resultant acts in the direction 4 0 upon the rigid body, it will hold the given forces in equilibrium. If it acts in the direction 0 4, it will replace the given forces.

We thus know the magnitude and direction of the resultant. But its *position* in the plane of the forces in Fig. 1 is as yet unknown.

In order to determine this, choose any point O in Fig. 2, and draw the lines $O0$ and $O4$. This point O we call the **pole** of the force polygon. Now since every line in the force polygon represents a force, by thus choosing a pole O and drawing lines $O0$, $O4$ to the extremities of the resultant 0 4, we have *resolved the resultant into* the two forces represented by $O0$ and $O4$. This is evident from the fact that these two lines make a closed polygon with 04, and hence taken as acting from 4 to O and O to 0, as shown by the arrows, hold the forces F_1, F_2, F_3, F_4 in equilibrium, or replace the resultant 4 0 (page 133). As the pole O is taken anywhere we please, we

can thus resolve the resultant 4 0 for equilibrium into forces in any two directions we wish.

Let us then consider the resultant 4 0 for equilibrium, replaced by the two forces 4*O* and *O*0. Anywhere in the plane of the forces in Fig. 1 draw a line s_0 parallel to *O*0 and produce it till it meets F_1, produced if necessary, at *a*.

If then we take s_0 and F_1, Fig. 1, as acting at *a*, their resultant will pass through *a* and be parallel to s_1 in the force polygon Fig. 2, because s_1 in the force polygon is the resultant of F_1 and s_0, since it closes the polygon for those forces. Through *a* in Fig. 1, then, draw a line parallel to s_1 and produce it to intersection *b* with F_2, produced if necessary. The line s_2 in the force polygon is the resultant of s_1 and F_2. Parallel to this line then draw s_2 through *b*, Fig. 1, and produce to intersection *c* with F_3, produced if necessary. The line s_3 in the force polygon is the resutlant of s_2 and F_3. Parallel to this line then draw s_3 through *c*, Fig. 1, and produce to intersection *d* with F_4, produced if necessary. Finally through *d* in Fig. 1 draw a line s_4 parallel to s_4 in the force polygon.

We thus find for any assumed position of s_0 in the plane of the forces in Fig. 1 the proper corresponding position of s_4. Since now s_0 and s_4 are components of the resultant in proper position and each may be considered as acting at any point in its line of direction, we have only to prolong them, and *their intersection gives a point e on the line of direction of the resultant.*

We prolong s_0 and s_4 then in Fig. 1 to intersection *e*. The line of direction of the resultant passes through *e*. Acting in the direction from 4 to 0, it will hold the forces in equilibrium. We thus know the magnitude, direction and position of the resultant for equilibrium.

Position of Pole and of s_0 Indifferent.—The method is evidently general no matter where in the plane of the forces in Fig. 1 we take s_0 as acting, and no matter where we take the pole in Fig. 2.

Pole, Equilibrium Polygon, Rays, Closing Line.—The point *O* we call the **pole** in the force polygon. It may be taken where we please. The polygon *abcd* in Fig. 1 we call the **equilibrium polygon**, and *ab*, *bc*, *cd*, etc., are its *segments*. In the present case it is evidently the shape a string would take if suspended at any two points as *A* and *B*, in Fig. 1, on s_0 and s_4. The stresses in the segments would be tensile. These stresses are given by the lines *O*0, *O*1, *O*2, in the force polygon, and we call these lines **rays**. In general forces may act up as well as down, in which case some of the segments would sustain compressive stresses and our equilibrium polygon would contain struts as well as ties.

Let us take any two points, as *A* and *B*, upon the end segments s_0 and s_4, Fig. 1, and suppose them fixed. The force s_0 acting at *A* we shall then have to *replace* by two forces, one parallel to the resultant and one in the direction *AB*. So also for s_4 at *B*. The sum of the two components parallel to the resultant must be equal and opposite to the resultant, and the component in the direction *AB* must be resisted by a strut or compression member *AB*. This resolution we make at once by drawing through *O* in the force polygon a line *OL* parallel to *AB*. The line *AB* we call the **closing line**. Thus we see from Fig. 2 that the sum of the components 4*L* and *L*0 equals the resultant.

In any case, then, we can fix any two points of the equilibrium polygon as *A*, *B*, by drawing the closing line *AB*. A line *OL* through *O* parallel to *AB*, in the force polygon, gives the components into which s_0 and s_4 are resolved.

We can then consider the entire polygon *AabcdB*, with its closing line *AB*, as *a frame in equilibrium* with the given forces, and can apply to it the principles of page 135.

Thus take the apex *A*. Here we have the reaction $R_1 = LO$ in equilibrium with the stresses in *AB* and *Aa*. Following round in the force polygon from *L* to 0, 0 to *O*, and *O* to *L*, and transferring these directions to the apex *A*, we find S_0 away from *A* or tension, and *OL* towards *A* or compression, just as on page 136.

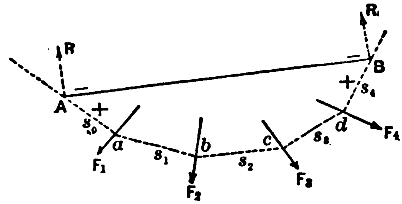


FIG. 1.

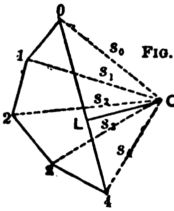


FIG. 2.

So also at the other apex *B* we have $R_2 = 4L$ in equilibrium with the stresses in *AB* and *Bd*. Following round in the force polygon from 4 to *L*, *L* to *O*, and *O* to 4, we find S_4 away from *B* or tension, and *LO* towards *B* or compression, as before. The components R_1 and R_2 act opposite to the resultant 04 which replaces the forces, and equal to it in magnitude. The forces at *A* and *B* parallel to *OL* are equal and opposite. Hence the frame is in equilibrium.

Recapitulation.—Our method, then, is as follows :

1st. Draw the force polygon by laying off the forces to scale one after the other, in any order. The line which closes this polygon gives the resultant in magnitude and direction. When it is taken as acting in the direction obtained by following round the force polygon in the direction of the forces, it will cause equilibrium. In the opposite direction it replaces the forces.

2d. Choose a pole *O*, and draw the rays s_0, s_1, s_2, etc.

3d. Draw the equilibrium polygon.

4th. Fix any two points in the end segments of the equilibrium polygon by drawing the closing line of the equilibrium polygon between those two points.

5th. A line drawn in the force polygon parallel to the closing line of the equilibrium polygon will divide the resultant into the two reactions at the ends. We thus have a frame the stresses in which can be found as on page 136.

Graphic Construction for Centre of Parallel Co-planar Forces.—Let F_1, F_2, F_3, etc., be parallel co-planar forces acting at the points A_1, A_2, A_3, etc., of a rigid body.

We construct the force polygon Fig. 2 by laying off the forces F_1, F_2, F_3, etc. The resultant is then the algebraic sum of the forces and parallel to them.

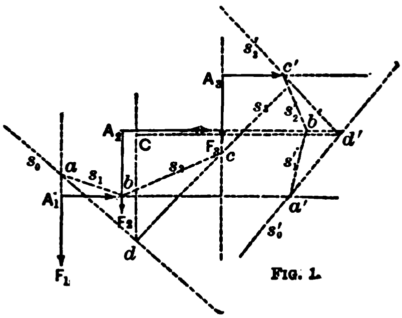
Then choose a pole *O* and draw the rays s_0, s_1, s_2, s_3, etc.

Anywhere in the plane of the forces, Fig. 1, we draw a line parallel to s_0 to intersection *a* with F_1; then *ab* parallel to s_1 to intersection *b* with F_2; then *bc* parallel to s_2 to intersection *c* with F_3; then s_3 through *c* parallel to s_3 in Fig. 2.

The intersection *d* of s_0 and s_3 is a point on the resultant which therefore has the direction and position *dC*.

Now suppose the forces F_1, F_2, F_3, etc., all turned in the same direction through a right angle.

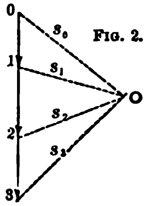
Draw the new equilibrium polygon $s_0'\, a'\, b'\, c'\, s_3'$, whose sides are respectively perpendicular to those of the first.



The intersection d' of s_0' and s_3' is a point on the resultant which therefore has the direction and position $d'C$.

The intersection C of the two resultants gives the centre of force for the system (page 73).

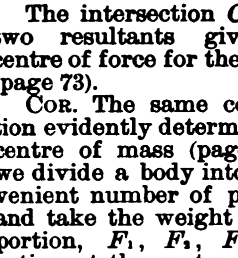
COR. The same construction evidently determines the centre of mass (page 75), if we divide a body into a convenient number of portions, and take the weight of each portion, F_1, F_2, F_3, etc., acting at the centre of mass of that portion.



Properties of the Equilibrium Polygon.—The equilibrium polygon has many interesting properties. We shall call attention to only two.

1st. As we have seen, the intersection of any two segments is a point in the resultant of the forces included between those segments. Thus in the preceding Fig. 1, the intersection d of s_0 and s_3 is a point on the resultant of F_1, F_2 and F_3

2d. Let s_0ab, Fig. 1, be a portion of the equilibrium polygon, and Fig. 2 its corresponding force polygon.



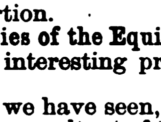
Take any line fe in Fig. 1, parallel to F_1 and draw the perpendicular $cd = x$.

Let $de = y$ be the ordinate between s_0 and s_1.

In the force polygon Fig. 2, draw the perpendicular $OH = H$ from the pole to 01. This is called the **pole distance** of F_1.

Then by similar triangles we have

$$y : x :: F_1 : H, \quad \text{or} \quad F_1x = Hy.$$



But F_1x is the moment of F_1 with reference to any point on the line fe.

Hence, *the moment of any force as F, with reference to any point, is equal to the ordinate through this point parallel to F_1, included between the segments of the equilibrium polygon which meet at F_1, multiplied by the pole distance of F_1 in the force polygon.*

Application to Parallel Forces.—The outer forces acting upon framed structures are generally weights and reactions of supports due to these weights. We have then in general to investigate a system of parallel forces,

Let F_1, F_2, F_3, Fig. 1, be vertical forces acting upon a rigid body or frame.

Lay off the force polygon 0 1 2 3, Fig. 2. Choose a pole O and draw the rays s_0, s_1, s_2, s_3.

Then in the plane of the forces Fig. 1, draw s_0 to meet F_1 at a; then s_1 through a to meet F_2 at b; then s_2 through b to meet F_3 at

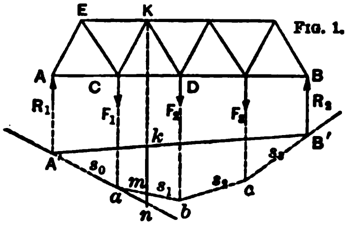


FIG. 1.

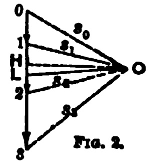


FIG. 2.

c; and finally s_3. We thus have the equilibrium polygon s_0abcs_3. We see that the horizontal component of the stress in any segment is constant and equal to OH (page 111).

Drop verticals through A and B which meet the end segments s_0 and s_3 in A' and B'. If we fix the points A', B' by drawing the closing line $A'B'$, the reactions at A', B' will be the reactions at A and B of the frame.

Therefore in Fig. 2, draw OL parallel to $A'B'$ and we have $L0 = R_1$, and $3L = R_2$.

Draw the pole distance OH. Through the apex K of the frame drop the vertical $Kkmn$. Then, as just proved, OH (to scale of force) $\times$ kn (to scale of distance) = the moment of R_1. Again, $OH \times mn$ = the moment of F_1. The resultant moment is then given by $OH \times (kn - nm)$ or $OH \times km$.

That is, for parallel forces, *the pole distance multiplied by the ordinate of the equilibrium polygon at any point, parallel to the forces included between the closing line and the polygon, gives the resultant moment of all the forces on either side of the ordinate with reference to any point in that ordinate.*

If then we make a section cutting EK, CK and CD, and take the centre of moments at K, we have (page 102) stress in CD $\times$ lever-arm for CD = algebraic sum of moments of R_1 and F_1 with reference to K. But this algebraic sum we have just seen is given by $H \times km$. Hence stress in CD is equal to $\dfrac{H \times km}{\text{lever-arm for } CD}$.

We can therefore find the moment graphically at any point by multiplying the ordinate to the equilibrium polygon at that point by the pole distance.

A few examples will make the application of the preceding principles clear.

Ex. 1. Let AB, Fig. 1, be a beam or rigid body or framed structure subjected to two unequal weights F_1 and F_2 applied at any two given points. Required the reactions at the supports A and B, also the moment at any point of all the forces right or left of that point, when equilibrium exists.

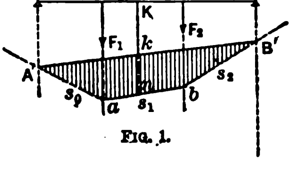


FIG. 1.

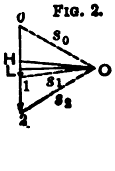


FIG. 2.

Draw the force polygon Fig. 2, choose a pole O, and draw s_0, s_1, s_2, and the pole distance H.

Construct the equilibrium polygon Fig. 1 by drawing a parallel to s_0 to intersection a

with F_1; through a a parallel to s_1 to intersection b with F_2; through b a parallel to s_2. Drop verticals from A and B and draw the closing line $A'B'$. Parallel to $A'B'$ draw OL in Fig 2.

Then $L0$ and $2L$ are the reactions at A and B; and since they act upwards, the supports must be below A and B.

The moment at any point K is equal to the ordinate kn multiplied by the pole distance H.

Ex. 2. It is well to observe that the order in which the forces are taken makes no difference in the results, although the figure obtained may be very different.

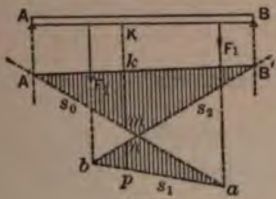


FIG. 1.

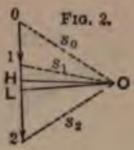


FIG. 2.

Thus take the same example as before, but number the forces in inverse order, Fig. 1.

We form the force polygon as before, choose a pole and draw s_0, s_1, s_2. Now parallel to s_0 we draw a line *till it meets F_1 at a* [note that s_0 must always be produced to meet F_1]; then from a a parallel to s_1 till it meets F_2 at b; then from b a parallel to s_2. Draw the closing line $A'B'$. A parallel to it in Fig. 2 gives the reactions $L0$ and $2L$ as before. At apex b of the equilibrium polygon we find s_2 tension, since F_2 acts downward. At apex a we find s_0 tension, since F_1 is downward. Hence at A', s_0 acts away from A', and following round in the force polygon we obtain $L0$ acting upwards. At B', s_2 acts away, and hence $2L$ acts upwards also. The supports at A and B must then be below.

As to the moments, the moment of the reaction at A with reference to any point K is $H \times km$. The moment of F_2 is $-H \times np$. The resultant moment is $H \times (Km - np)$. The lower ordinates subtracted from the upper will give us the same figure as before.

Whenever, then, we obtain a double figure as in the present case, it shows that we have taken the forces in inconvenient order. We have only to change the order to obtain the moments directly from the equilibrium polygon.

Closing Line at Right Angles to the Forces—Choice of Pole Distance.—It makes no difference what inclination the closing line may have, because, as we have seen, the ordinate in the equilibrium polygon parallel to the resultant, multiplied by the pole distance, gives the resultant moment, *with reference to any point on that ordinate*, of all the forces right or left.

We can, however, if we wish always cause the closing line to be at right angles to the parallel forces. We have only to find first by preliminary construction the reactions or the point L. If then we take a new pole anywhere in a line through this point at right angles to the forces, the closing line will be at right angles to the forces.

As to choice of pole distance, we have only to so choose the position of the pole as to give good intersections for the polygon.

The multiplication may be directly performed by properly changing the scale in the equilibrium polygon. The ordinate to this new scale will then give the moment at once. Thus if our scale of length in Fig. 1, preceding, is five feet to an inch, and the pole distance in the force polygon Fig. 2, measured to the scale of force adopted, is ten pounds, we have only to take fifty moment units to an inch as the scale for the ordinates and they will give the moments directly.

Ex. 3. Let the single weight F_1 act at any point of the rigid body AB. Then the equilibrium polygon is $A'aB'$. The vertical reactions at A and B are $L0$ and $1L$, both acting up, and hence the supports are below A and B.

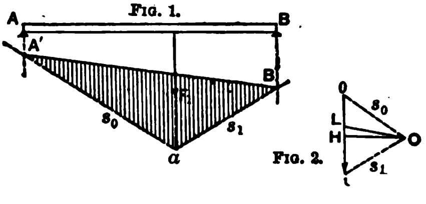


FIG. 1. FIG. 2.

We see at once that the moment is greatest at the weight and decreases to zero at each support.

Ex. 4. Let F_1 act outside of the supports A and B. Observe in

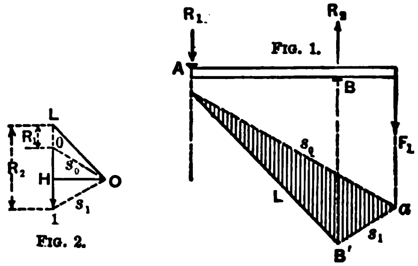


FIG. 1. FIG. 2.

constructing the equilibrium polygon that s_0 is always produced till it meets F_1; also that the closing line $A'B'$ always unites the two points vertically under the supports, upon the two end segments.

The reactions require special notice. Thus the reaction R_2 at B is the resultant of the stresses in aB' and $B'A'$, or $1L$ in the force polygon. The reaction R_1 at A is the resultant of the stresses in $A'a$ and $A'B'$, or $L0$ in the force polygon.

Since F_1 acts downward at apex a, we have s_1 compression and s_0 tension. Therefore at apex A' we take s_0 acting away, and hence obtain $L0$ acting down, or the support is above A.

At apex B' we take s_1 acting towards, and hence obtain $1L$ acting up, or the support is below B.

Ex. 5. ONE DOWNWARD AND ONE UPWARD FORCE BETWEEN THE SUPPORTS.—Here we need only call special attention to the fact that as F_2 acts up and is less than F_1, s_2 in the force polygon Fig. 2 lies between s_0 and s_1.

The reaction at A is the resultant of s_0 and L or $L0$. The reaction at B is the resultant of s_2 and L or $L2$. Since F_1 is down at a, we have s_0 tension, and since F_2 is up at b, we have s_2 tension. At apex A', then, s_0 acts away, and hence L is compression and $L0$ acts upward and support at A is below. At apex B', s_2 acts away, and L is compression as before and $2L$ acts downward, or support at B is above.

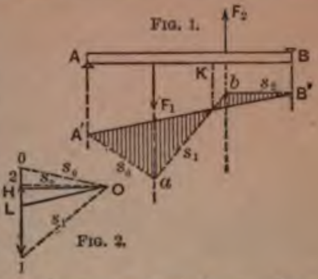


FIG. 1. FIG. 2.

We see also that if F_2 were less, so that 2 falls below L in the force polygon, the reaction at B would be upward also, and the support would then have to be below. The student should sketch the case for F_2 greater than F_1.

At the point K we see that the moment is zero. If AB is a beam, the point K is the "point of inflection," or the point at which the curve of deflection of the beam changes from concave to convex. The beam would be concave upwards as far as K, and from there on convex upwards.

Ex. 6. In the preceding case, let the forces be equal. Laying off the force polygon Fig. 2, the first force extends from 0 to 1, and the second from 1 back to 0. Choosing a pole O and drawing s_0, s_1, s_2, we find that s_0 and s_2 coincide.

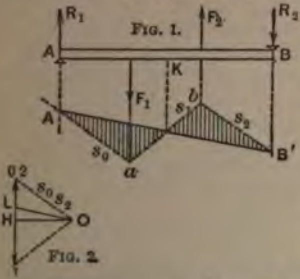


FIG. 1. FIG. 2.

Constructing the equilibrium polygon and drawing the closing line $A'B'$ and its parallel L in the force polygon, we see that the reaction at A or the resultant of s_0 and L is $L0$, and the reaction at B or the resultant of s_2 and L is also $L0$. The reactions are therefore equal. Since s_0 and s_2 are both tension, we have reaction at A upward or support below A, and reaction at B downward or support above B.

This is in accord with the principle (page 73) that a couple can only be held in equilibrium by another couple. Morever, the resultant of s_0 and s_2 in Fig. 2 is zero, and the point of application is at the intersection of s_0 and s_2 in Fig. 1, or at an infinite distance.

That is, the resultant of a couple is zero at an infinite distance (page 73.)

At K the moment is zero as before, and we have a point of inflection.

Ex. 7. TWO EQUAL WEIGHTS BEYOND THE SUPPORTS.—The figure needs no explanation, except to call attention to the reactions.

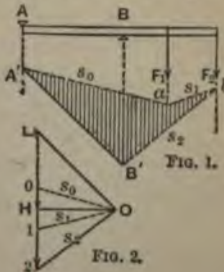


FIG. 1. FIG. 2.

Thus the reaction at A is $L0$ acting down. At B it is $2L$ acting up.

The moment at any point, in all cases, is the ordinate multiplied by the pole distance H. The shaded areas then show how the moments vary.

We repeat here that the order in which the forces are taken, in all cases, as also the position of the pole, is indifferent. The student will do well to work out cases to scale and satisfy himself that this is true.

EX. 8. TWO EQUAL AND OPPOSITE FORCES BEYOND THE SUPPORTS.

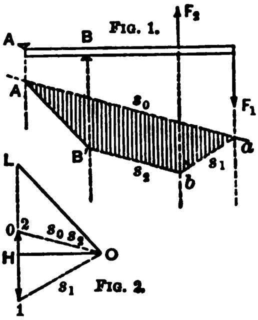


FIG. 1. FIG. 2.

—Observe that s_0 is produced till it intersects F_1 at a in Fig. 1; then s_1 from a to b; then s_2 parallel to s_2 or s_0 in Fig. 2. The closing line $A'B'$ is then drawn. A parallel to it in Fig. 2 gives L.

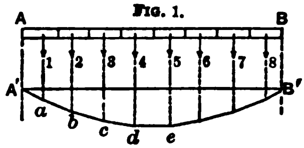
The reaction at A is $L0$ acting down, and at B, $0L$ acting up.

Between B and F_2 the moment is constant. This is the graphic interpretation of the principle, page 72, that the moment of a couple is constant for any point in its plane.

EX. 9. A UNIFORMLY-DISTRIBUTED LOAD. —Let the load be uniformly distributed. We might consider it as a system of equal and equidistant weights very close together.

Thus in Fig. 1 the load area, which is a rectangle of uniform density, whose height is the load per unit of length, and whose length is AB, may be divided into any number of equal parts. The weight on each of these parts acts at its centre of mass. We can then lay off the force polygon Fig. 2. Since the reactions at A and B are equal, we take the pole in a horizontal through the middle point of the force line. The closing line $A'B'$ will then be parallel to AB (p. 149). We can then draw s_0, s_1, s_2, etc., and construct the equilibrium polygon. It is evident that the points a, b, c, d, etc., will enclose a curve tangent to ab, bc, cd, etc., at the points midway between, that is, where the lines of division of the load area meet the sides of the equilibrium polygon.

FIG. 1.



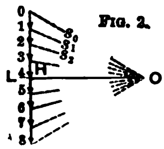


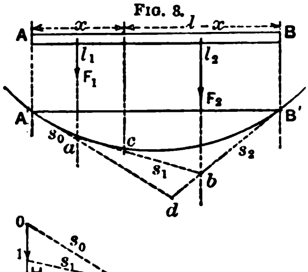
FIG. 2.

The ordinates to this curve, multiplied by the pole distance H, give the moment at any point on the ordinates.

It will be seen, however, that this method is deficient in accuracy, because the lines ab, bc, cd, etc., are so short and there are so many of them. If, however, we can find what the curve $A'abcd$, etc., is, we could draw the curve at once.

Suppose we divide the load area into only two portions of lengths x and $l-x$, where $l = AB$, Fig. 3. The entire weight over the portion x can be considered as acting at the centre e_1 of the load area. The same holds good for the portion $l - x$. We thus have two forces F_1 and F_2.

FIG. 3.



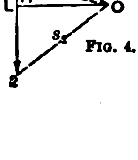


FIG. 4.

Taking the pole as before, so that the closing line $A'B'$ shall be parallel to AB, construct the equilibrium polygon $A'abB'$. The curve of moments will be tangent at A', c and B', as shown by the dotted curve.

Now we see that, no matter where the load area is supposed to be divided, we shall always have for the distance $e_1 e_2$ between F_1 and F_2

$$e_1e_2 = \frac{1}{2}x + \frac{1}{2}\left(l - x\right) = \frac{1}{2}l.$$

That is, no matter where the line of division is taken, the horizontal projection of the line ab of the equilibrium polygon is constant and equal to $\frac{1}{2}l$. But ab is a tangent to the curve required. But if from any point on the line $A'd$ we draw a line ab limited by the line $B'd$, so that the horizontal projection is constant, the line ab will *envelop a parabola.*

This may easily be proved as follows: Let the load per unit of length be p. Then the entire load is pl and the reaction at each end is $\frac{pl}{2}$.

The moment at any point distant x from the left support is then

$$y = \frac{pl}{2}x - F_1\frac{x}{2}.$$

But since F_1 is equal to px,

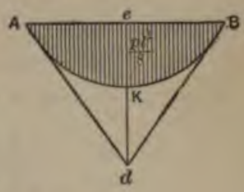
$$y = \frac{pl}{2}x - \frac{px^2}{2}.$$

This is the equation of a parabola. At the centre $x = \frac{l}{2}$, and we therefore have the centre ordinate $\frac{pl^2}{8}$.

COR. 1. We see, therefore, that when a string is suspended from two points A', B' and sustains a load uniformly distributed over the horizontal, the curve of equilibrium is a parabola (page 113).

Also the horizontal component of the stress at any point, as is evident from the force polygon, is constant and equal to H. Also the vertical component of the stress at any point as c, Fig. 3, is $R_1 - F_1$, or equal to the total load between the lowest point and the point considered (page 111).

COR. 2. We have the following construction for the equilibrium curve. Lay off a perpendicular eK at the centre e and make it equal by scale to $\frac{pl^2}{8}$. Through A, K and B construct a parabola having its vertex at K. The ordinate to this parabola through any point will give the moment at that point.



The distance Kd is also equal to $\frac{pl^2}{8}$, because the moment of the reaction with reference to e is

$$ed = \frac{pl}{2} \times \frac{l}{2} = \frac{pl^2}{4},$$

and $Kd = ed - eK = \frac{pl^2}{4} - \frac{pl^2}{8} = \frac{pl^2}{8}$.

COR. 3. *How to Draw a Parabola.*—Since we know, then, the distance $ed = \frac{pl^2}{4}$, we can always draw the lines Ad and Bd. If

then we divide Ad and Bd into any number of equal parts and number these parts along one line away from d and along the other towards d, we have only to draw lines joining any two points having the same number and these lines will all have the same horizontal projection $\frac{l}{2}$.

They will therefore enclose the parabola required. Tangent to these lines we may sketch the curve.

A better method is to plot the ordinates to the curve from its equation,

$$y = \frac{pl}{2}x - \frac{px^2}{2}.$$

Methods of Solution of Framed Structures.—In Chap. IV we have given and illustrated two methods of computation for framed structures:

1st. By Resolution of Forces (page 101).

2d. By Moments or the "Method by Sections" (page 102).

In the present Chapter we have the corresponding graphic methods:

1st. By Resolution of Forces (page 135).

2d. By Moments (147).

EXAMPLES.

(1) *A roof-truss has a span of* 50 *ft. and a centre height of* 12.5 *ft. Each rafter is divided into four equal panels, and the lower horizontal tie is divided into six equal panels. The bracing is as shown in the figure. Find the stresses in the members, by the graphic method of moments, for a weight of* 800 *lbs. at each upper apex.*

Ans. We have computed the stresses (page 105, Ex. (8)) by the two methods resolution of forces and moments. We have also found the stresses by the graphic method of resolution of forces (page 140, Ex. (1)).

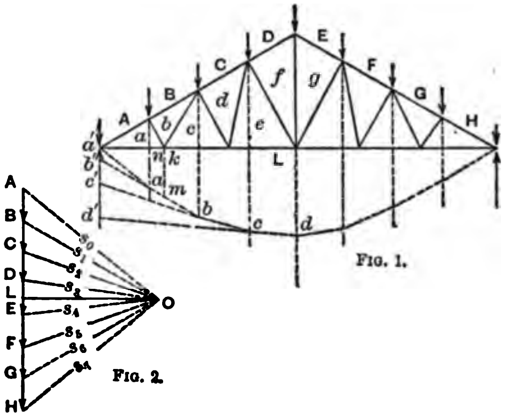


FIG. 1.

FIG. 2.

We can construct the force polygon Fig. 2, and then the equilibrium polygon Fig. 1. This, however, is not advisable for reasons already given. It will be more accurate to assume the pole distance as unity, thus discarding the force polygon altogether, and construct points in a parabola from the equation

$$y = \frac{pl}{2}x - \frac{px^2}{2}.$$

In the present case the load per foot is, if we suppose half weights of 400 at the ends, $\frac{6400}{50} = 128$ lbs. $= p$. Taking $x = \frac{1}{8}l, \frac{2}{8}l$, etc., we have

$$x = \frac{1}{8}l, \quad \frac{2}{8}l, \quad \frac{3}{8}l, \quad \frac{4}{8}l;$$

$$y = 17500 \quad 30000 \quad 37500 \quad 40000 \text{ lb.-ft.}$$

Laying these off to any convenient scale, we determine very accurately the points a, b, c, d of the equilibrium polygon. The other half of the polygon is precisely similar.

The ordinates to this polygon will give, to the scale adopted, the moment, for any point of the truss, of the outer forces left or right. Thus the moment with reference to k of all forces right or left is km, Fig 1. We find by scale $km = 21666\frac{2}{3}$ lb.-ft. In the same way for the next lower apex we find the moment 35000 lb.-ft. The moment at the next lower apex or centre of the span is 40000 lb.-ft.

Now by the method of sections (page 102) we have for any member

$$\text{Stress} \times \text{lever-arm} + \Sigma \text{ moments of outer forces} = 0.$$

The second term is given by the ordinates of the equilibrium polygon to scale.

As regards the centre of moments for any member, we must observe the rule (page 102), viz: Cut the truss entirely through by a section cutting only three members the strains in which are unknown. For any one of these take the point of moments at the intersection of the other two.

For the proper sign for the first member of the equation place an arrow on the cut member pointing away from the end belonging to the left-hand portion, and take the moment (+) or (−) according as the rotation indicated by this arrow is counter-clockwise or clockwise.

If the stress comes out positive, it indicates tension; if negative, compression.

Take for instance the first lower panel, La. The centre of moments must be taken at the first upper apex. The moment for this point is given by the ordinate na of the equilibrium polygon, or − 17500 lb.-ft. We take the minus sign, because the rotation is clockwise. We have then

$$La \times 3.125 - 17500 = 0, \quad \text{or} \quad La = +\,5600 \text{ lbs.},$$

where 3.125 ft. is the lever-arm of La.

In similar manner we have

$$Lc \times 6.25 - 30000 = 0, \quad \text{or} \quad Lc = +\,4800 \text{ lbs.},$$

where 6.25 ft. is the lever-arm of Lc.

For Le we have

$$Le \times 9.375 - 37500 = 0, \quad \text{or} \quad Le = +\,4000 \text{ lbs.},$$

where 9.375 ft. is the lever-arm of Lc.

For the first upper panel Aa, take the centre of moments at k. The moment for this point is given by the ordinate from *k to the first line of the polygon produced.* It is therefore larger than km, which gives the combined moment of the reaction and first weight. We find it by scale to be $-\,23333\frac{1}{3}$ lb.-ft.

We have then

$$-\,Aa \times 3.727 - 2333\tfrac{1}{3} = 0, \quad \text{or} \quad Aa = +\,6260 \text{ lbs.},$$

where 3.727 ft. is the lever-arm for Aa.

In like manner for Bb we have centre of moments at k, and moment $km = -\,21666\frac{2}{3}$. Hence

$$-\,Bb \times 3.727 - 21666\tfrac{2}{3} = 0, \quad \text{or} \quad Bb = +\,5818 \text{ lbs.}$$

For Cd we have

$$-Cd \times 7.454 - 35000 = 0, \quad \text{or} \quad Cd = +4691 \text{ lbs.},$$

where 7.454 ft. is the lever-arm for Cd.

For Df we have

$$-Df \times 11.151 - 40000 = 0, \quad \text{or} \quad Df = +3587 \text{ lbs.}$$

For all the braces the point of moments is at the left-hand end. Taking a section through Bb, ab and La, we have acting on the left-hand portion only the weight AB and the reaction. The moment of the weight relative to the left end is the ordinate $a'b'$, or by scale -5000 lb.-ft. The lever-arm for ab is 6.934 ft. Hence

$$-ab \times 6.934 - 5000 = 0, \quad \text{or} \quad ab = -721 \text{ lbs.}$$

For bc we have

$$+ab \times 6.934 - 5000 = 0, \quad \text{or} \quad ab = +721 \text{ lbs.}$$

For cd the moment is $a'b' + b'c'$, or -15000. We have then

$$-cd \times 13.869 - 15000 = 0, \quad \text{or} \quad cd = -1081 \text{ lbs.},$$

and so on. All lever-arms can be scaled off the frame or must be computed.

The present method is not to be recommended for the braces. In prolonging the sides ab, bc, etc., of the equilibrium polygon, a slight variation in direction will make considerable error in the ordinate at the end. Also as the sides ab, bc, etc., are short they do not give direction accurately enough.

Of all our four methods, the graphic method by resolution of forces (page 135) is the easiest of application to such cases.

The more irregular the frame the more advantageous it is.

(2) *A bridge-girder, as shown in the figure,* 10 *feet deep,* 80 *feet long, eight equal panels in the lower chord and seven equal panels in the upper chord, has a load of* 5 *tons at each lower apex. Find the stresses by diagram and by moments.*

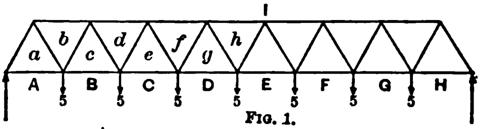


FIG. 1.

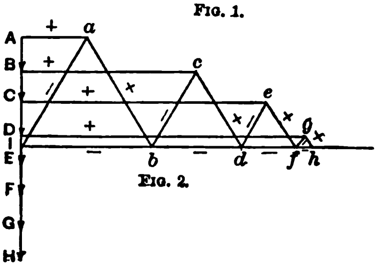


FIG. 2.

Ans. The panel length is 10 ft., sec $\theta = 1.117$. By moments then

$Aa \times 10 - 17.5 \times 10 = 0,$ or $Aa = +17.5$ tons.

$Bc \times 10 - 17.5 \times 15 + 5 \times 5 = 0,$ $Bc = +23.75$ "

$$Ce \times 10 - 17.5 \times 25 + 5\,(5 + 15) = 0, \qquad Ce = +\,33.75 \text{ tons.}$$
$$Dg \times 10 - 17.5 \times 35 + 5(5 + 15 + 25) = 0, \qquad Dg = +\,38.75 \quad \text{"}$$
$$-\,Ib \times 10 - 17.5 \times 10 = 0, \qquad Ib = -\,17.5 \quad \text{"}$$
$$-\,Id \times 10 - 17.5 \times 20 + 5 \times 10 = 0, \qquad Id = -\,30 \quad \text{"}$$
$$-\,If \times 10 - 17.5 \times 30 + 5(10 + 20) = 0, \qquad If = -\,37.5 \quad \text{"}$$
$$-\,Ih \times 10 - 17.5 \times 40 + 5(10 + 20 + 30) = 0, \qquad Ih = -\,40 \quad \text{"}$$

$$Ia = -\,17.5 \times 1.117 = -\,19.55, \qquad de = -\,7.5 \times 1.117 = -\,8.38,$$
$$ab = +\,19.55, \qquad ef = +\,8.38,$$
$$bc = -\,12.5 \times 1.117 = -\,13.96, \qquad fg = -\,2.5 \times 1.117 = -\,2.79,$$
$$cd = +\,13.96, \qquad gh = +\,2.79.$$

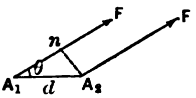
CHAPTER VII.

WORK.

WORK INDEPENDENDENT OF PATH. UNIT OF WORK. VIRTUAL DISPLACEMENT. VIRTUAL WORK. PRINCIPLE OF VIRTUAL WORK.

Work.—The product of a uniform force by the projection of the displacement of its point of application along the line of action of the force is called *work*.

Thus let a uniform force, that is, a force constant in direction and magnitude, act at a point A_1, and let the displacement of the point of application be $A_1A_2 = d$.



Let θ be the angle FA_1A_2 between the force and the displacement. Then the projection of the displacement $A_1A_2 = d$ upon the line of the force F is $A_1n = d\cos\theta$, and we have for the work W,

$$W = Fd\cos\theta. \qquad (1)$$

But $F\cos\theta$ is the projection of the force F upon the line of the displacement A_1A_2.

Hence, *work is the product of a constant force by the projection of the displacement of its point of application along the line of the force, or the product of the displacement by the projection of the force along the line of the displacement.*

If the projection of the displacement A_1n along the force is in the direction of the force, the force is said to *do work*. In this case the angle θ is acute and W in equation (1) is positive.

If the projection of the displacement A_1n along the force is opposite in direction to the force, work is said to *be done against the force*. In this case the angle θ is obtuse and W in equation (1) is negative.

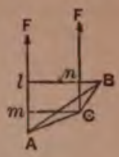
Cor. 1. If the displacement is at right angles to the constant force, the work is zero.

Cor. 2. The weight of a body is a force acting at the centre of mass (page 76). Hence the work done against gravity in raising a body of mass m through a distance s is $W = -mgs$, where mg is the weight in poundals and s the displacement of the centre of mass. In gravitation units (page 6), $W = -ms$.

Cor. 3. The work done by gravity upon a body of mass m which falls through a distance s is $W = +mgs$, where mg is the weight in poundals and s the displacement of the centre of mass. In gravitation units, $W = +ms$.

Work Independent of the Path.—The definition for work given in the preceding Article evidently holds good no matter what the path, provided the force is uniform, that is, does not change in direction or magnitude.

Thus let the constant force F act on the particle A which is displaced from A to B either along the line AB, or from A to C and from C to B.



In the first case the work is $F \times Al$. In the second case the work is

$$F \times Am + F \times Cn = F \times Al.$$

So in general for any broken line between A and B. Since a curve is the limit of a polygon, the same holds true for any curved path between A and B.

Work when Force is Variable.—If the force is variable, we must take the displacement indefinitely small, so that the force during such displacement may be considered as uniform. In such case we have

$$W = \int F ds. \qquad (2)$$

Unit of Work.—If $[F]$ is the unit of force and F the number of units of force, $[L]$ the unit of distance and s the number of units of distance in the direction of the force, $[W]$ the unit of work and W the number of units of work, we have

$$W[W] = F[F] \times s[L].$$

We have then the numeric equation

$$W = Fs,$$

provided

$$[W] = [F] \times [L].$$

The unit of work, then, is the work done by one unit of force when the displacement in the direction of the force is one unit of distance.

The English absolute unit of work is then the *foot-poundal*, or a constant force of one poundal acting through one foot.

The C. G. S. absolute unit of work is a constant force of one dyne acting through one centimeter. It is called an *erg*. A multiple of this, equal to 10000000 ergs or 10^7 ergs, is used in electrical measurements and called a *joule*, after Dr. James Prescott Joule.

In English gravitation units (page 6) the unit of work is the *foot-pound*. This is the unit commonly adopted in Engineering calculations. It is the work done in raising a mass of one pound through the vertical distance of one foot against gravity. It is therefore a variable amount of work, since the weight of one pound varies with the locality (page 6).

Virtual Displacement—Virtual Work.—When the point of application of a force is actually displaced, the displacement is *actual* and the work done by or against the force is *actual* also.

If F is the force acting at any point and s is the actual displacement in the direction of the force of that point, then if F remains uniform, that is, constant in magnitude and direction during the displacement, then the actual work is Fs.

But in general, when the point of application of a force is displaced, the force does not remain uniform unless the displacement is taken indefinitely small.

If F, then, is the force acting at any point and ds is an indefinitely small displacement in the direction of the force, we have in general the work given by Fds.

Now an indefinitely small displacement of a point which does

not actually take place, but which is only imagined or supposed to take place, we may distinguish by calling a ***virtual displacement***, and we call the work done by or against a force by reason of the virtual displacement of its point of application the **virtual work** of the force.

Virtual displacement unless otherwise specified is always to be taken as indefinitely small. It is always ***linear displacement***, since a point has no size.

Principle of Virtual Work.—Let F_1, F_2, F_3, etc., be any number of concurring forces, that is, forces acting upon a particle at P, and suppose this particle to receive a virtual displacement PD in any direction.

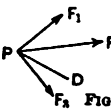


Fig. 1.

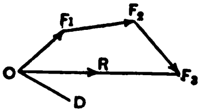


Fig. 2.

Since virtual displacement is indefinitely small, the forces remain unchanged in direction and magnitude.

If we lay off the line representatives in Fig. 2, the resultant is given in magnitude and direction by the closing line $OF_3 = R$ of the force polygon.

Draw OD parallel and equal to PD, and let α_1, α_2, α_3, etc., and θ be the angles made by F_1, F_2, F_3, etc., and R with OD.

Then we have by construction

$$R\cos\theta = F_1\cos\alpha_1 + F_2\cos\alpha_2 + F_3\cos\alpha_3, \text{ etc.} = \Sigma F\cos\alpha.$$

That is, the component of the resultant R in the direction of the virtual displacement is equal to the algebraic sum of the components of the forces in that direction.

If we multiply by the displacement $PD = d$, we have

$$R.d\cos\theta = F_1.d\cos\alpha_1 + F_2.d\cos\alpha_2 + F_3.d\cos\alpha_3, \text{ etc.} = \Sigma Fd\cos\alpha.$$

But since d is indefinitely small, so that the forces remain unchanged in magnitude and direction, we have by definition $R.d\cos\theta$ equal to the virtual work of the resultant, and $F_1.d\cos\alpha_1$, $F_2.d\cos\alpha_2$, etc., equal to the virtual works of F_1, F_2, etc.

Hence, *if a particle acted upon by any system of forces receive a virtual displacement in any direction whatever, the algebraic sum of the virtual works of the forces is equal to the virtual work of the resultant.*

If the forces F_1, F_2, F_3, etc., acting on the particle are in equilibrium, their resultant R is zero, and we have

$$F_1.d\cos\alpha_1 + F_2.d\cos\alpha_2 + F_3.d\cos\alpha_3, \text{ etc.} = \Sigma Fd\cos\alpha = 0.$$

This is called the "principle of virtual work"; a principle which includes all of statics and kinetics. We may state it as follows:

If a particle in equilibrium under the action of any system of forces receive a virtual displacement in any direction whatever, the algebraic sum of the virtual works of the forces is equal to zero.

Conversely, *if the algebraic sum of the virtual works of a system of forces acting on a particle is zero for every virtual displacement whatever, the particle is in equilibrium.*

COR. 1. If a system of particles is in equilibrium under the action of external and internal forces, and any number of particles of the system receive any virtual displacement whatever, then, since the algebraic sum of the virtual works of the forces acting on each particle is zero, it follows that the algebraic sum of the virtual works of all the forces, external and internal, is zero.

The principle of virtual work applies then to any material system if all forces external and internal are considered.

COR. 2. If a system of particles in equilibrium under the action of external and internal forces receive any virtual displacement *of translation* whatever which *does not alter the configuration of the system*, then no work is done by or against the internal forces, and the algebraic sum of the virtual works of the external forces alone is zero.

The principle of virtual work applies then to the external forces acting upon any rigid body in equilibrium, if the body is regarded as a particle and the virtual displacement is one of translation.

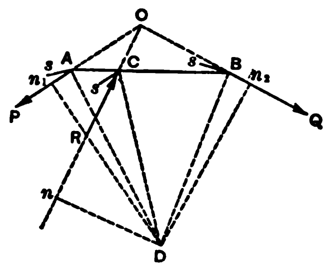
COR. 3. If a system of rigid bodies in equibrium under the action of external and internal forces receive any virtual displacement whatever which does not alter the configuration of the system, then no work is done by or against the internal forces, and the algebraic sum of the virtual works of the external forces alone is zero.

The principle of virtual work applies then to the external forces acting upon any system of rigid bodies whose configuration does not change, if the rigid bodies are regarded as particles and their virtual displacements are translations.

EXAMPLES.

(1) *A lever ACB with fulcrum at C is acted upon by the co-planar forces P and Q at the ends A and B. Find the conditions for equilibrium, neglecting friction.* (For rough lever see Ex. (17), page 221.)

Ans. Let R be the resultant acting at the fulcrum C.



Take any point D in the plane of the forces. Let the lever be rotated counter-clockwise about an axis through D at right angles to the plane of the forces, through an *indefinitely small* angle of θ radians. Then the virtual displacement of A is $\overline{AD}\,.\,\theta = As$, making the angle $sAP = \alpha_1$ with P. The virtual displacement of C is $\overline{CD}\,.\,\theta = Cs$, making the angle $sCR = \alpha$ with the resultant R. The virtual displacement of B is $\overline{BD}\,.\,\theta = Bs$, making the angle $sBO = \alpha_2$ with the direction of Q.

Then by the principle of virtual work, having regard to the proper signs as given by the figure,

$$+P\,.\,As\cos\alpha_1 - R\,.\,Cs\cos\alpha - Q\,.\,Bs\cos\alpha_2 = 0,$$

or

$$+P\,.\,\theta\,.\,\overline{AD}\cos\alpha_1 - R\,.\,\theta\,.\,\overline{CD}\cos\alpha - Q\,.\,\theta\,.\,\overline{BD}\cos\alpha_2 = 0,$$

or

$$+P\,.\,\overline{AD}\cos\alpha_1 - R\,.\,\overline{CD}\cos\alpha - Q\,.\,\overline{BD}\cos\alpha_2 = 0.$$

But if we drop from D the perpendiculars $Dn_1 = p$ on P, $Dn = r$ on R, and $Dn_2 = q$ on Q, we have $\overline{AD}\cos\alpha_1 = p$, $\overline{CD}\cos\alpha = r$, $\overline{BD}\cos\alpha_2 = q$, and hence

$$+Pp - Rr - Qq = 0.$$

That is, the algebraic sum of the moments of the forces about any point in their plane is zero (page 99).

Again, suppose the lever to be translated in any direction through an indefinitely small distance, so that the virtual displacement of every point is d, and let the forces P, Q, R make the angles α_1, α_2 and α with the direction of the displacement. Then by the principle of virtual work we have

$$Pd \cos \alpha_1 + Qd \cos \alpha_2 + Rd \cos \alpha = 0,$$

or

$$P \cos \alpha_1 + Q \cos \alpha_2 + R \cos \alpha = 0.$$

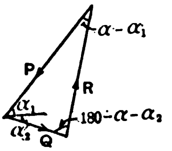
That is, the algebraic sum of the components of the forces in any direction is zero (page 99), and their line representatives make a closed polygon.

Again, since the algebraic sum of the moments about any point is zero, the three forces must intersect at a common point O (page 100).

If we suppose the fulcrum C to be fixed, we can have only rotation. We can then easily prove by the principle of virtual work that the necessary and sufficient condition of equilibrium for *any body* free to turn about a fixed axis under the action of *any number* of forces is that the algebraic sum of the moments of the external forces with reference to the fixed axis shall be zero.

If we take the fulcrum C as our point of moments we easily deduce, as on page 71, *when the forces are parallel,*

$$R = P + Q, \quad \frac{P}{Q} = \frac{BC}{AC}.$$



If the forces are not parallel, let the force P make the angle α_1, the force Q the angle α_2, the force R the angle α, with the lever, the acute values being taken.

Then since the line representatives form a closed polygon, we have

$$P : Q :: \sin (180 - \alpha - \alpha_2) : \sin (\alpha - \alpha_1),$$

or

$$\frac{P}{Q} = \frac{\sin \alpha \cos \alpha_2 + \cos \alpha \sin \alpha_2}{\sin \alpha \cos \alpha_1 - \cos \alpha \sin \alpha_1}.$$

We have also

$$R \sin \alpha = P \sin \alpha_1 + Q \sin \alpha_2 ;$$

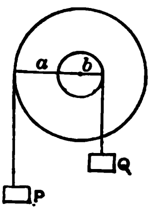
$$R \cos \alpha = P \cos \alpha_1 - Q \cos \alpha_2 ;$$

$$\tan \alpha = \frac{P \sin \alpha_1 + Q \sin \alpha_2}{P \cos \alpha_1 - Q \cos \alpha_2};$$

$$R^2 = P^2 + Q^2 - 2PQ \cdot \cos (\alpha_1 + \alpha_2).$$

(2) *In a wheel and axle the radius of the wheel is a, and of the axle b. Find the conditions for equilibrium, neglecting friction and rigidity of the rope, when a mass P hung from the wheel just balances a mass Q hung from the axle.* (For friction and rigidity see Ex. (18), page 222.)

Ans. The external forces are Pg and Qg. If we suppose P to receive a virtual displacement s downward, then Q will receive the virtual displacement $\frac{b}{a}s$ upward, and by the principle of virtual work we have



$$Pgs - Qg \cdot \frac{b}{a}s = 0, \quad \text{or} \quad Pa = Qb,$$

or the algebraic sum of the moments of the external forces with reference to the fixed axis is zero. This is the sole condition for equilibrium for any body free to turn about a fixed axis acted upon by any number of forces.

In this example we see that it is not necessary to suppose the virtual displacement indefinitely small, since the forces do not vary with the displacement.

(3) *Four sailors, each exerting a force of* 112 *lbs., can just raise an anchor by means of a capstan whose radius is* 1 *foot* 2 *in. and whose spokes are* 8 *ft. long, measured from the axis. Find the weight of the anchor.*

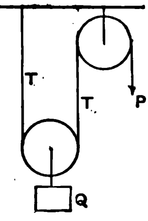
Ans. 3072 lbs.

(4) *If the length of each of a pair of sculls be* 8 *ft.* 6 *in., and the distance from the hand to the rowlock be* 2 *ft.* 3 *in., find the force on the boat when the rower applies a force of* 25 *lbs. on each scull, assuming that the blade does not move through the water.*

Ans. 68 lbs.

(5) *In the single movable pulley find the relation between the force P and the mass Q for equilibrium, disregarding friction and rigidity of the rope.* (For friction and rigidity see Ex. (19), page 224.)

Ans. The external forces are P and the weight of the mass Q. If we suppose a virtual displacement of Q downward equal to s, the corresponding virtual displacement of P will be $2s$ upward. We have then by the principle of virtual work, in gravitation units,

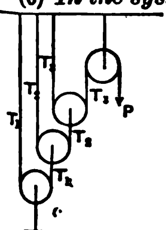


$$Qs - 2Ps = 0, \quad \text{or} \quad P = \frac{Q}{2}.$$

Again, if T is the tension of the rope, we have, in gravitation units, $T = P$ and $2T = Q$. Hence $P = \frac{Q}{2}$.

In this example we see that it is not necessary to suppose the virtual displacement indefinitely small, because the forces do not vary with the displacement.

(6) *In the system of pulleys shown, find the relation between the force P and the mass Q for equilibrium, disregarding friction and rigidity of the ropes.* (For friction and rigidity see Ex. (20), page 225.)



Ans. The external forces are P and the weight of the mass Q. If we suppose a virtual displacement of Q downward equal to s, the displacement of the next pulley is $2s$, of the next $4s$, and so on. If there are n movable pulleys, then, each one of the mass m, we have by the principle of virtual work, in gravitation units,

$$Qs + m.s + m.2s + m.4s + \ldots m.2^{n-1}s - P.2^n s = 0.$$

Hence

$$P = \frac{Q + m(1 + 2 + 2^2 + 2^3 + \ldots 2^{n-1})}{2^n},$$

or

$$P = \frac{Q + (2^n - 1)m}{2^n}.$$

If we disregard the mass m of the pulleys,

$$P = \frac{Q}{2^n}.$$

In this example we see it is not necessary to suppose the virtual displacement indefinitely small, because the forces do not vary with the displacement.

Again, let the tensions of the ropes be $T_1, T_2, \ldots T_n$. Then we have for equilibrium, in gravitation units,

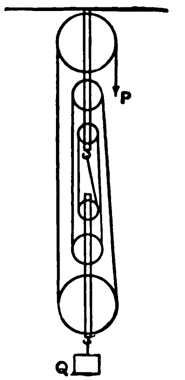
$$2T_1 = Q + m;$$
$$2T_2 = T_1 + m;$$
$$2T_3 = T_2 + m;$$
$$2T_n = T_{n-1} + m;$$
$$P = T_n.$$

Multiplying the second equation by 2, the next by 2^2, the next by 2^3, etc., and the last by 2^{n-1} and adding, we have as before

$$2^nP = Q + m + 2m + 2^2m + 2^3m + \ldots 2^{n-1}m.$$

(7) *In the system of pulleys shown, find the relation between P and Q for equilibrium, disregarding friction.* (For friction and rigidity see Ex. (21), page 225.)

Ans. The external forces are P and the weight of Q. If we suppose a virtual displacement of Q downward equal to s, each string coming from the lower block will be lengthened by s, and the virtual displacement of P upwards will be ns, where n is the number of strings coming from the lower block. We have then by the principle of virtual work, if m is the mass of the lower block,



$$(Q + m)s - nsP = 0,$$

or

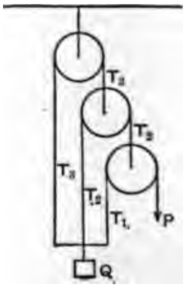
$$P = \frac{Q + m}{n}.$$

In this example we see that it is not necessary to suppose the virtual displacement indefinitely small, because the forces do not vary with the displacement.

Again, the tension in each string is the same and equal to P. Hence, if n is the number of strings coming from the lower block, $nP = Q + m$.

(8) *In the system of pulleys shown, find the relation between P and Q for equilibrium, disregarding friction and rigidity of the ropes.* (For friction and rigidity see Ex. (22), page 226.)

Ans. The external forces are P and the weight of Q. If we suppose a virtual displacement of Q downward equal to s, then the highest movable pulley will be raised a distance s, the next will be raised twice the height through which the highest is raised plus the distance through which Q descends, that is, through the distance $3s$.



In the same way any movable pulley will rise through the height s plus twice the distance through which the pulley next above rises.

If the number of pulleys is n and the mass of each pulley is m, the distances through which each pulley is raised are respectively s, $(2^2 - 1)s$, $(2^3 - 1)s \ldots (2^{n-1} - 1)s$. Also P will be moved vertically upwards a distance $(2^n - 1)s$. We have then by the principle of virtual work in gravitation units,

$$Qs - m(2 - 1)s - m(2^2 - 1)s - m(2^3 - 1)s \ldots - m(2^{n-1} - 1)s - (2^n - 1)Ps = 0.$$

Hence

$$P = \frac{Q - m[(2-1)+(2^2-1)+(2^3-1)\ldots+(2^{n-1}-1)]}{2^n-1}$$

or

$$P = \frac{Q + mn - (2^n-1)m}{2^n-1}.$$

If we neglect the mass of the pulleys,

$$P = \frac{Q}{2^n-1}.$$

In this example we see that it is not necessary to suppose the virtual displacement indefinitely small, because the forces do not vary with the displacement.

Again, if n is the number of pulleys and T_1, T_2, T_3, etc., the tensions in the strings, then we have for equilibrium, in gravitation units,

$$T_1 = P; \quad \ldots \ldots \quad (1)$$

$$T_2 = 2T_1 + m; \quad \ldots \ldots \quad (2)$$

$$T_3 = 2T_2 + m; \quad \ldots \ldots \quad (3)$$

$$T_n = 2T_{n-1} + m; \quad \ldots \ldots \quad (n)$$

$$T_1 + T_2 + T_3 + \ldots T_n = Q. \quad \ldots \ldots \quad (4)$$

Multiplying the second equation by 2^{n-1}, the third by 2^{n-2}, the nth by 2, and adding, we have

$$2T_n = 2^nP + 2^{n-1}m + 2^{n-2}m + \ldots 2m.$$

Adding equations (2), (3), . . . (n) and employing equation (4), we have

$$Q - P = 2(Q - T_n) + (n-1)m.$$

Eliminating T_n, we have, as before,

$$2^{n-1}P = Q - (2-1)m - (2^2-1)m - (2^3-1)m \ldots - (2^{n-1}-1)m.$$

(9) *If we have three movable pulleys arranged as in Example* (6), *their masses, beginning with the lowest, being* 9, 3 *and* 1 *lb. respectively, find what force P will support a mass of* 69 *lbs.*

Ans. 11 lbs.

(10) *If in the system of Example* (7) *there are nine pulleys and each has a mass of one pound, find the force P to support a mass of* 85 *lbs.*

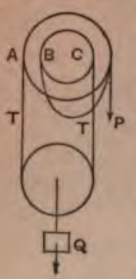
Ans. 9 lbs.

(11) *If in the system of Example* (8) *the mass supported is* 56 *lbs., and each movable pulley, of which there are three, has a mass of* 1 *lb., find the horizontal distance of the centre of mass of Q from the centre of the fixed pulley when the diameters of all the pulleys are equal.*

Ans. Nine twenty-eighths the radius of the pulley.

(12) *In the differential pulley shown in the figure an endless chain passes over a fixed pulley A, then under a movable pulley to which the mass Q is attached, and then over another fixed pulley B, a little smaller but coaxial with A. The two pulleys A and B are in one piece and obliged to turn together through the same angle. The two ends of the chain are joined so as to form a loop. The force P is applied to the right-hand portion of the loop. To prevent the chain from slipping, there are cavities in the circumferences of the upper*

pulleys into which the links of the chain fit. Find the relation of P to Q for equilibrium, neglecting friction. (For friction see Ex. (23), page 227.)



Ans. Let b be the radius of the pulley B, and a the radius of the pulley A.

Let Q receive a virtual displacement vertically downwards equal to s. Then, since both A and B turn through the same angle θ, we have

$$\frac{a\theta - b\theta}{2} = s, \quad \text{or} \quad \theta = \frac{2s}{a-b},$$

and P has the virtual displacement vertically upwards of $a\theta = \dfrac{2as}{a-b}$.

We have then by the principle of virtual work, in gravitation units,

$$Qs - P \cdot \frac{2as}{a-b} = 0, \quad \text{or} \quad P = \frac{Q(a-b)}{2a}.$$

In this example it is not necessary to suppose s indefinitely small, because the forces do not vary with the displacement. Again, let T be the tension of the chain. Then if the pulley is in equilibrium, we have in gravitation units

$$2T = Q.$$

Taking moments about the axis of C,

$$Ta - Tb - Pa = 0.$$

Hence

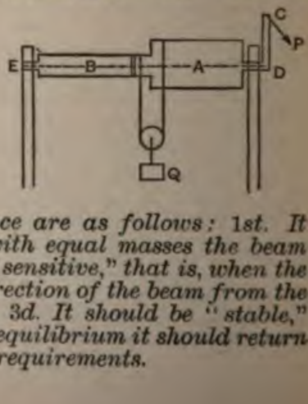
$$P = \frac{Q(a-b)}{2a}.$$

By taking a and b nearly equal we can have P as small as we please.

(13) *In the differential wheel and axle shown in the figure, we have two axles B and A of different radii, rigidly connected and turning about their common axis DE. The force P is applied at right angles to the axis at the extremity of the arm CD. The mass Q is attached to a pulley supported by a rope which is wrapped one way round B and the other way round C. Find the relation of P to Q for equilibrium, neglecting friction.*

Ans. Let c be the arm CD, and b and a be the radii of B and A. Then, as in the preceding example,

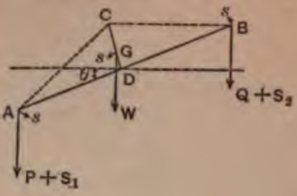
$$P = \frac{Q(a-b)}{2c}.$$



By taking b and a nearly equal we can have P as small as we please. In the simple wheel and axle the same result can only be obtained by making c inconveniently large or a inconveniently small.

(14) *The requisites of a good balance are as follows:* 1st. *It should be "true," that is, when loaded with equal masses the beam should be horizontal.* 2d. *It should be "sensitive," that is, when the masses differ by a small quantity the direction of the beam from the horizontal should be easily perceptible.* 3d. *It should be "stable," that is, when moved from its position of equilibrium it should return to it quickly. Show how to secure these requirements.*

Ans. Let the masses of the loads be P and Q, and of the scale-pans S_1 and S_2. Let G be the centre of mass of the balance, not including the scale-pans, W its mass. Let C be the point of support, and let CG be perpendicular to the beam AB at D. Let θ be any angle of the beam with the horizontal, and denote CD by h, CG by k, and let $AD = a$, and $BD = b$.



Suppose the balance rotated through an indefinitely small angle $d\theta$ about D. Then the virtual displacement of A is $As = ad\theta$; of B, $Bs = bd\theta$; of G, $Gs = (h-k)d\theta$.

We have then, by the principle of virtual work,

$$(P+S_1)ad\theta\cos\theta - (Q+S_2)bd\theta\cos\theta + W(h-k)d\theta\sin\theta = 0.$$

If we take moments about D, we have for equilibrium also,

$$(P+S_1)a\cos\theta - (Q+S_2)b\cos\theta + W(h-k)\sin\theta = 0.$$

Hence

$$\tan\theta = \frac{(Q+S_2)b-(P+S_1)a}{W(h-k)}.$$

1st. When the loads are equal, $P = Q$ and $S_1 = S_2$. In order, then, that the balance may be "true," that is, $\theta = 0$ when the loads are equal, we must have $a = b$. The arms must therefore be equal. We have then for a true balance, when the masses of the scale-pans are equal,

$$\tan\theta = \frac{(Q-P)a}{W(h-k)}. \qquad (1)$$

We can easily test the truth of a balance by interchanging the loads which hold the beam horizontal. If the beam settles again into a horizontal position, since the loads are equal the balance is true.

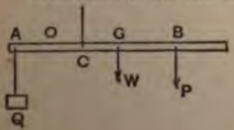
It is almost impossible to make a balance perfectly true. When, therefore, great accuracy is required, the method of *double weighing* is adopted. This enables us to determine the exact mass, however untrue the balance. It consists in first making the beam horizontal with the body whose mass is required in one scale and sand or shot in the other. Then the body is replaced by known masses sufficient to keep the beam horizontal.

2d. From equation (1) we see that if a true balance is to be "sensitive," that is, if θ is to be large when $Q - P$ is small, we must have $h - k$ small with reference to a. That is, the distance GD of the centre of mass G from the beam must be small compared to the length of arm. This requisite is then obtained by making a large and bringing the centre of mass near the beam.

3d. But we see from the figure that when k is large the moment Wk of W about the point of support C is large and the balance will return more readily then when k is small. The condition of "stability" then requires that the distance GD of the centre of mass G from the beam shall be large. The conditions of stability and sensitiveness are then at variance.

In scientific measurements, where great accuracy is required, the third requisite is sacrificed to obtain the second, and time is required. For ordinary commercial purposes, where it is desirable to save time, the reverse is the case.

(15) *Show how to graduate the common steelyard.*



Ans. Let P be the movable weight, W the weight of beam and scale-pan acting at the centre of mass G, Q the weight to be determined at A, all in gravitation units. Let C be the point of suspension. Let n be the number of the graduation at B, so that $Q = nP$. We have then for equilibrium

$$nP \times \overline{AC} - W \times \overline{CG} - P \times \overline{CB} = 0.$$

If we put $n = 0$ in this equation, we obtain the position O of the zero of the scale,

$$\overline{CO} = -\frac{W}{P}\overline{CG},$$

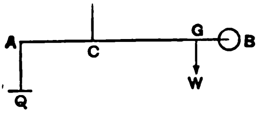
or O is on the other side of C to W at a distance $\frac{W}{P}\overline{CG}$ from it. Hence

$$nP \times \overline{AC} = P \times \overline{OB}, \quad \text{or} \quad \overline{OB} = n\overline{AC}.$$

The graduations are obtained, then, by marking off distances from O equal to $\overline{AC}$, $2\overline{AC}$, $3\overline{AC}$, etc. Intermediate graduations correspond to fractional values of n.

(16) *Show how to graduate the Danish steelyard.*

Ans. This steelyard consists of a beam AB terminating in a heavy ball B.



From the end A hangs the scale-pan. The fulcrum C is moved until the weight of the mass in the scale-pan is balanced by that of the steelyard. Let Q be the mass at A, W the mass of steelyard and scale-pan acting at the centre of mass G. Evidently the zero of graduation is at G, since the beam balances when the fulcrum is there, when there is no mass Q.

We have $Q = nW$, and for equilibrium

$$nW \times AC = W \times CG = W(AG - AC)$$

Hence

$$AC = \frac{AG}{n+1}.$$

The graduations then are at distances from A equal to $\frac{AG}{2}, \frac{AG}{3}, \frac{AG}{4}$, etc.

(17) *If the arms of a false balance are horizontal when there are no weights in the scale-pans and one arm is one ninth part longer than the other, and if in using it the substance to be weighed is put as often into one scale as into the other, show that the seller loses five ninths per cent on his transactions.*

(18) *If a common steelyard is* 18 *inches long, weighs* 3 *lbs. and is suspended at a point* 3 *inches from one extremity, what is the greatest mass which can be measured by a movable weight of* 2 *lbs.?*

Ans. 16 lbs.

(19) *If one arm of a common balance be longer than the other, show that the real weight of the body is the geometrical mean between its apparent weights as weighed first in one scale and then in the other.*

(20) *The arms of a false balance are unequal and one of the scales is loaded. A body whose true mass is P lbs. appears to weigh Q lbs. when placed in one scale and Q′ lbs. when placed in the other. Find the ratio of the arms and the weight with which the scale is loaded.*

Ans. $\frac{Q' - P}{P - Q}, \quad \frac{QQ' - P^2}{P - Q}.$

CHAPTER VIII.

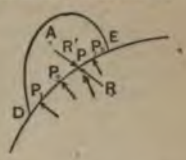
CONSTRAINED EQUILIBRIUM—SMOOTH CURVE OR SURFACE.

REACTION OF A CURVE OR SURFACE. REACTION OF A SMOOTH CURVE OR SURFACE. EQUILIBRIUM OF A BODY ON A SMOOTH CURVE OR SURFACE. EQUILIBRIUM OF A BODY AT ANY POINT OF A SMOOTH CURVE OR SURFACE. GENERAL EQUATIONS.

Reaction of a Curve or Surface.—When a particle is in contact with a rigid material curve or surface, the force or pressure which the curve or surface exerts upon the particle is called the **reaction** of the curve or surface.

If then we introduce this reaction as an additional force in combination with all the other forces acting upon the particle, we can remove the curve or surface and consider the particle by itself as acted upon by this reaction and all the other forces.

Equilibrium of a Body on Any Curve or Surface. — Let a rigid body ADE rest in equilibrium upon a rigid material curve of surface DE, smooth or rough, and touch it at many points P_1, P_2, P_3, etc.



Let the reactions at these points be R_1, R_2, R_3, etc., and let the resultant reaction be R acting at the point P of the curve or surface. If all the reactions are *pressures* exerted by the curve or surface upon the body, this point P must evidently *always lie within the line or surface of contact DE.*

Since all reactions are internal to the system composed of the body and curve or surface, they are *internal forces* or *stresses* (page 7) and the resultant reaction R is the *resultant stress*. All other forces acting upon the body are external to the system, and we call them, therefore, *external forces.*

Now if the body is in equilibrium on the curve or surface, the resultant R' of all the external forces must be equal and opposite to the resultant reaction R and lie in the same straight line. Its line of direction must therefore pass through P.

This point P, if the curve or surface resists by pressure only, must always lie within the line or surface of contact DE.

If the base DE is a point, or the body touches the curve or surface at a single point only, the body is in equilibrium at this point, the line of direction of R' must pass through this point and R' must be equal and opposite to R at this point.

If the line of direction of R' falls outside the base DE the body

will rotate. If it intersects the curve or surface in the perimeter of the base, as at E, the body is said to be in **limiting stability.**

If we consider all stresses but one as external forces, the body may be treated as a particle at the point of application of this one. *Whenever, then, we speak of a body as "in equilibrium at any point of a curve or surface," the point referred to may be any one of the points of contact with the curve or surface. The body may be treated as a particle of equal mass placed at this point.*

Reaction of a Smooth Curve or Surface.—When a particle is in equilibrium upon any curve or surface, the reaction must be equal and opposite to the resultant of all the external forces.

If the curve or surface is perfectly smooth, it can offer no resistance to a tangential force acting upon the particle.

The reaction and the resultant of all the external forces must then, for equilibrium, not only be equal and opposite, but must also be normal to the curve or surface. For if the resultant of all the external forces is not normal, it can be resolved into a normal and a tangential component. But the smooth curve or surface can offer no resistance to the tangential component. Hence for equilibrium the resultant of all the external forces must be normal and the equal and opposite reaction must also be normal.

A smooth curve or surface, then, is one whose reaction is normal. It is incapable of offering resistance to motion in any other than a normal direction.

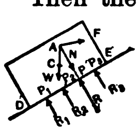
Equilibrium of a Body on a Smooth Curve or Surface.—As we have just seen, whether the curve or surface be smooth or rough, we can treat the body as a particle of equal mass placed at any one of the points of contact with the curve or surface.

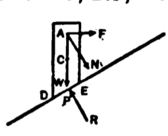
If the curve or surface is smooth, then, as we have just seen, the reactions R_1, R_2, R_3, etc., at each and every point of contact must each be normal at its own point of application, the resultant reaction R must be normal at P, and the resultant R' of all the external forces must be normal and its line of direction must pass through P.

If the curve or surface resists by pressure only, this point P must lie within the line or surface of contact.

Thus, for example, let a body ADE rest in equilibrium on a smooth plane surface DE.

Then the reactions R_1, R_2, R_3, etc., at every point of contact P_1, P_2, P_3, etc., are normal to the plane. The resultant reaction R is normal to the plane also and acts at some point P of the base DE. If the surface resists by pressure only, this point P must lie within the base DE.





Let W be the weight of the body acting vertically at the centre of mass C, and let F be the resultant of all the other external forces. The resultant N of W and F is then the resultant of all the external forces. It must pass through the intersection A of W and F, and if there is equilibrium must be equal and opposite to the resultant reaction R and lie in the same straight line. It must therefore also be normal to the plane, and its line of direction must intersect the plane at the same point P of the base DE. If N falls outside of the base DE, there is no equilibrium if the plane resists by pressure only. If N passes through E, the body is in limiting stability.

We can consider the body as a particle placed at any one of the points of contact.

[Equilibrium of a Body at Any Point of a Smooth Curve or Surface.]—If a body acted upon by any number of forces F_1, F_2, etc., applied at different points, is at rest at any point of a smooth curve or surface, we may then treat it as a particle placed at that point. The normal reaction N at the point must be equal and opposite to the resultant of all the other forces acting upon the body. The curve or surface can then be replaced by its normal reaction N at the point.

The normal to a surface at any point has a definite direction. The normal to a curve at any point may have any direction in a plane through that point perpendicular to the tangent at that point.

Let all the forces acting upon the body, *not including the normal reaction N at the point P*, be F_1, F_2, etc., making with the co-ordinate axis the angles $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, etc. Then the components parallel to the axes are

$$F_x = F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + \ldots = \Sigma F \cos \alpha;$$
$$F_y = F_1 \cos \beta_1 + F_2 \cos \beta_2 + \ldots = \Sigma F \cos \beta;$$
$$F_z = F_1 \cos \gamma_1 + F_2 \cos \gamma_2 + \ldots = \Sigma F \cos \gamma.$$

1. Equilibrium of a Body at Any Point of a Smooth Curve.—Let ds be an element of the curve. Then the direction-cosines of the tangent to the curve at any point P given by the co-ordinates (x, y, z) are $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$. The normal reaction N at the point P has no component tangent to the curve at this point. If all the other forces are resolved along the tangent to the curve at this point, the sum of their tangential components is $F_x\frac{dx}{ds} + F_y\frac{dy}{ds} + F_z\frac{dz}{ds}$. If there is equilibrium, this sum must be zero.

We have then for the condition of equilibrium

$$F_x\frac{dx}{ds} + F_y\frac{dy}{ds} + F_z\frac{dz}{ds} = 0. \qquad (1)$$

If we multiply by ds, we obtain

$$F_x dx + F_y dy + F_z dz = 0,$$

which is the *principle of virtual work* (page 159).

2. Equilibrium of a Body at Any Point of a Smooth Surface.—Let the normal reaction N at the point P make with the co-ordinate axes the angles θ_x, θ_y, θ_z. Then we have for equilibrium

$$\left.\begin{array}{c} F_x = N \cos \theta_x, \quad F_y = N \cos \theta_y, \quad F_z = N \cos \theta_z; \\ F_x^2 + F_y^2 + F_z^2 = N^2. \end{array}\right\} \qquad (a)$$

Let the equation of the surface be $u = 0$, where u is a function of x, y, z.

For convenience of notation let

$$\frac{du}{dx} = U, \quad \frac{du}{dy} = V, \quad \frac{du}{dz} = W, \quad \text{and} \quad U^2 + V^2 + W^2 = Q^2.$$

Then the direction-cosines of the normal to the surface at the point (x, y, z) are

$$\left.\begin{aligned}
\cos\theta_x &= \frac{U}{Q} = \frac{\frac{du}{dx}}{\sqrt{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2}};\\
\cos\theta_y &= \frac{V}{Q} = \frac{\frac{du}{dy}}{\sqrt{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2}};\\
\cos\theta_z &= \frac{W}{Q} = \frac{\frac{du}{dz}}{\sqrt{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2}}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But for equilibrium

$$N = \frac{F_x}{\cos\theta_x} = \frac{F_y}{\cos\theta_y} = \frac{F_z}{\cos\theta_z}.$$

We have then by inserting the values for θ_x, θ_y, θ_z,

$$\frac{F_x}{\left(\frac{du}{dx}\right)} = \frac{F_y}{\left(\frac{du}{dy}\right)} = \frac{F_z}{\left(\frac{du}{dz}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we substitute the values of $\cos\theta_x$, $\cos\theta_y$, $\cos\theta_z$ in equations (*a*), and multiply the first equation by dx, the second by dy, the third by dz, then add the results and reduce by the equation

$$\left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy + \left(\frac{du}{dz}\right)dz = 0,$$

which is the total differential of the equation of the surface $u = 0$, we obtain

$$F_x dx + F_y dy + F_z dz = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which is the *principle of virtual work* (page 159).

Equations (3) give two independent simultaneous equations which combined with the equation of the surface will determine the point of equilibrium, if there be one. Equation (4) is the condition of equilibrium.

If all the forces are in one plane, let this be the plane of XY. Then from equations (3) and (4), since $F_z = 0$ and $dz = 0$,

$$\frac{F_x}{\left(\frac{du}{dx}\right)} = \frac{F_y}{\left(\frac{du}{dy}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

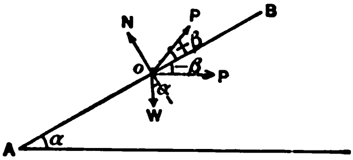
$$F_x dx + F_y dy = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

EXAMPLES.

(1) *A body of weight W is placed upon a smooth inclined plane AB which makes an angle α with the horizontal and is acted upon*

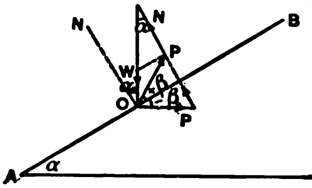
by a force P which makes the angle β with the plane. Find the conditions of equilibrium. (For rough plane see Ex. 7, page 215.)

Ans. Consider the body as a particle placed at any point O on the plane (page 169). We have acting upon the particle the weight W, the force P and the normal reaction N of the plane, and these three forces must constitute a system of concurrring forces in equilibrium.



Let the angle $BOP = \beta$ be positive when above the plane and negative when below the plane, as shown in the figure.

1st Solution : By Resolution of Forces.—If we lay the line representatives of the forces off in order the same way round, they form a triangle (page 62).



We have then directly

$$N : W :: \sin [90 - (\alpha + \beta)] : \sin (90 + \beta),$$

or

$$N = \frac{W \cos (\alpha + \beta)}{\cos \beta}. \quad . \quad . \quad (1)$$

$$P : W :: \sin \alpha : \sin (90 + \beta),$$

or

$$P = \frac{W \sin \alpha}{\cos \beta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We see at once from the figure that when $\beta = +(90° - \alpha)$, P and W are equal in magnitude and act opposite in direction and N is zero. For any greater value of positive β, N is negative and there is no equilibrium possible.

For negative β, we must evidently have β less than $90°$.

Equations (1) *and* (2) *hold good, then, for all values of* β *between* $+(90° - \alpha)$ *and* $-90°$. Outside of these limits there is no equilibrium.

The *minimum value* of P is for $\beta = 0$ and equal to $P = W \sin \alpha$.

Again, we can put the algebraic sum of the components along the plane and perpendicular to the plane equal to zero (page 61). We have then

$$N + P \sin \beta - W \cos \alpha = 0;$$
$$P \cos \beta - W \sin \alpha = 0.$$

From these two equations we obtain the same equations (1) and (2) for N and P.

Again, we can put the algebraic sum of the horizontal and vertical components equal to zero.

Hence

$$P \sin (\alpha + \beta) + N \cos \alpha - W = 0;$$
$$P \cos (\alpha + \beta) - N \sin \alpha = 0.$$

From these two equations we obtain the same equations (1) and (2) for N and P.

2d Solution : By Virtual Work.—In order to find P, suppose a virtual displacement d *along the plane from O towards B*. This displacement is at right angles to N and hence the virtual work of N is zero.

For equilibrium the algebraic sum of the virtual works of P, N and W is equal to zero.

The component of P in the direction of the displacement is $P \cos \beta$. The virtual work of P is then $+ Pd \cos \beta$. The component of W on the line of the displacement is $W \sin \alpha$ opposite in direction to the displacement. The virtual work of W is then $- Wd \sin \alpha$. The virtual work of N is zero. Hence

$$Pd \cos \beta - Wd \sin \alpha = 0, \quad \text{or} \quad P = \frac{W \sin \alpha}{\cos \beta}.$$

In order to find N, we might suppose a virtual displacement at right angles to P, thus making the virtual work of P zero. Since, however, P is now known, let us suppose a *horizontal* virtual displacement d away from O. Then the virtual work of W is zero, and we have

$$Pd\cos(\alpha+\beta) - Nd\sin\alpha = 0.$$

Hence

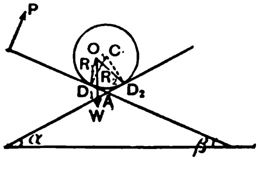
$$N = \frac{P\cos(\alpha+\beta)}{\sin\alpha} = \frac{W\cos(\alpha+\beta)}{\cos\beta}.$$

In this example we see it is not necessary to suppose the virtual displacements indefinitely small, because the forces do not vary with the displacement.

(2) *A body of weight W is placed in contact with the under side of a smooth inclined plane which makes an angle α with the horizontal, and is acted upon by a force P which makes an angle β with the plane. Find the conditions of equilibrium.* (For rough plane see Ex. (8), *page* 217.)

Ans. $N = -\frac{\cos(\beta+\alpha)}{\cos\beta}W$, $P = \frac{W\sin\alpha}{\cos\beta}$, where $\beta > +(90-\alpha)$ and $< +90$.

(3) *Find the force P necessary to just move a cylinder of radius r and weight W up a plane inclined at an angle α, by a crowbar of length l inclined at an angle β, neglecting friction.* (For friction see Ex. (9), page 218.)



Ans. The weight W acting at the centre O can be resolved into components N_1, N_2 perpendicular to the bar and plane. If P acts at right angles to the bar, we have by virtual work, for a small displacement due to turning the bar about A through an indefinitely small angle θ,

$$Pl\theta - N_1 . \overline{AN_1}\theta = 0, \quad \text{or} \quad P = \frac{N_1 . \overline{AN_1}}{l}.$$

But

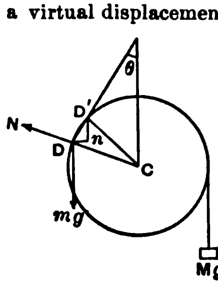
$$\overline{AN_1} = r\tan\frac{1}{2}\left(\alpha+\beta\right) = \frac{r[1-\cos(\alpha+\beta)]}{\sin(\alpha+\beta)}, \quad \text{and} \quad N_1 = \frac{W\sin\alpha}{\sin(\alpha+\beta)}.$$

Hence

$$P = \frac{Wr\sin\alpha[1-\cos(\alpha+\beta)]}{l\sin^2(\alpha+\beta)} = \frac{Wr\sin\alpha}{l[1+\cos(\alpha+\beta)]}.$$

(4) *A particle of mass m rests on a smooth cylinder and is kept in equilibrium by a string fastened to another particle of mass M, which passes over the cylinder and hangs freely. Determine the position of equilibrium.* (For rough cylinder see Ex. (10), page 218.)

Let the position of equilibrium be at D and suppose a virtual displacement DD' along the chord at D. Then M moves through a distance equal to the chord DD' and we have the algebraic sum of the virtual works zero, or, since the virtual work of N is zero,



$$Mg \times \text{chord } DD' - mg \times nD' = 0,$$

$$\text{or} \quad \frac{M}{m} = \frac{nD'}{\text{chord } DD'}.$$

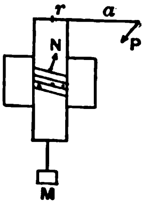
If DD' is indefinitely small, it is tangent at D. Hence if the tangent at D makes an angle θ with the vertical, we have for the condition of equilibrium

$$\frac{M}{m} = \cos\theta.$$

In this example we see that the condition of an indefinitely small virtual displacement is necessary, because the forces vary with the displacement.

(5) *Find the conditions for equilibrium for a screw, neglecting friction.* (For friction see Ex. (11), page 219.)

Ans. Let P be the force applied at the end of the arm a, and let the pitch of the screw or distance between the threads be p. Let M be the mass supported by P.



1st. *By Virtual Work.*—If the arm a moves through 2π radians, M is raised the distance p. If it moves through one radian, M is raised $\frac{p}{2\pi}$.

If P, then, has a virtual displacement of θ radians, it moves through the distance $a\theta$ and M is raised a distance $\frac{p\theta}{2\pi}$, and we have by the principle of virtual work, in gravitation units,

$$Pa\theta - \frac{Mp\theta}{2\pi} = 0, \quad \text{or} \quad P = \frac{Mp}{2\pi a}.$$

Hence

$$\frac{M}{P} = \frac{2\pi a}{p} = \frac{\text{circumference of circle in which } P \text{ moves}}{\text{distance between threads}}.$$

2d. *By Resolution of Forces.*—Let N be the normal pressure on each thread, and α the inclination of the thread to the horizontal. Then, in gravitation units, we have for equilibrium

$$\Sigma N \cos\alpha - M = 0.$$

If r is the radius of the screw, we have, taking moments about the axis, for equilibrium

$$-Pa + \Sigma N \sin\alpha \times r = 0.$$

But if the screw be developed, we have an inclined plane whose base is $2\pi r$ and height p and angle of inclination α.

Therefore

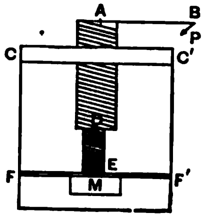
$$2\pi r \tan\alpha = p, \quad \text{or} \quad 2\pi r \sin\alpha = p\cos\alpha.$$

Inserting this value of $r\sin\alpha$, we have, as before,

$$-Pa + \frac{Mp}{2\pi} = 0, \quad \text{or} \quad P = \frac{Mp}{2\pi a} = \frac{Mr\tan\alpha}{a}.$$

(6) *The differential screw consists of a screw AD which works in a fixed nut CC'. AD is hollow and has a thread cut inside, in which a solid screw DE works. DE is prevented from turning by some means, for instance by a rod FEF' rigidly connected with it, whose ends work in grooves, so that DE can only move in a direction parallel to its axis. The mass M is raised by the force P, applied at the end of the arm AB = a. Find the condition of equilibrium, neglecting friction.* (For friction, see Ex. (12), page 220.)

Ans. Let a be the length of arm AB, P the force applied, p and p' the pitch of screws AD and DE.



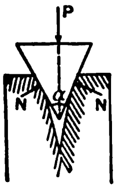
When AB turns through 2π radians, AD rises a distance p. DE cannot turn and therefore moves downwards a distance p' relatively to AD. The mass M is raised, then, a distance $p - p'$. When AB turns through one radian, M is raised $\frac{p-p'}{2\pi}$. If P then has a virtual displacement of θ radians, it moves through the distance $a\theta$ and M is raised $\frac{(p-p')\theta}{2\pi}$.

Hence by the principle of virtual work, in gravitation units,

$$Pa\theta - M\frac{(p-p')\theta}{2\pi} = 0, \quad \text{or} \quad P = M\frac{p-p'}{2\pi a}.$$

Evidently, by making p and p' nearly equal, we can make P as small as we please. In the simple screw the same result is attained only by making the lever-arm a inconveniently large, or by making the pitch so small that the thread is too weak to support the pressure on it.

(7) *Let the force acting normally upon the middle of the back of an isosceles wedge be P. Find the conditions for equilibrium, neglecting friction.* (For friction see Ex. (13), page 220.)



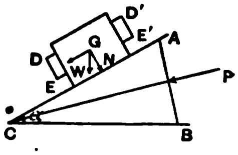
Ans. The pressure on each side must be normal. Let α be the angle of the wedge. Then for a virtual displacement of s we have by the principle of virtual work

$$Ps - 2Ns\sin\frac{\alpha}{2} = 0, \quad \text{or} \quad P = 2N\sin\frac{\alpha}{2}.$$

(8) *Let an isosceles wedge rest with its surface BC upon a horizontal plane. Let a force P be applied normally at the middle point of the back. Let the body, whose weight is W, acting at the centre of mass G, rest upon the wedge, and be constrained by guides DE, D'E' to move in a direction normal to AC. Find the condition for equilibrium, neglecting friction.*

Ans. Let α be the angle of the wedge. Then $N = W\cos\alpha$.

$$P = 2W\cos\alpha\sin\frac{\alpha}{2}.$$



(9) *A body weighing 10 lbs. rests on a smooth plane rising 2 feet vertically for every 5 ft. along the plane. It is kept from sliding by a force in the direction of the plane. Find the force and the pressure on the plane.*

Ans. $P = 4$ lbs.; $N = 9.16$ lbs.

(10) *A body is kept at rest on a smooth inclined plane by a force acting up the plane equal to half the weight of the body. Find the inclination of the plane.*

Ans. 30°.

(11) *A body is at rest on a smooth inclined plane, and the applied force and pressure on the plane are each equal to the weight of the body. Find the inclination of the plane and the direction of the applied force.*

Ans. 60°; 30° to inclined plane and horizontal plane.

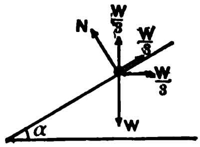
(12) *A body is supported on a smooth inclined plane by a force equal to its weight. Show that the reaction of the plane is double what it would be if the body were supported by the least possible force.*

(13) *Let P be the force which, acting up a smooth inclined plane, keeps a body in equilibrium. Let Q be the force which supports the body when its direction is such that it is equal to the reaction of the plane. Show that P acting up the plane could just support a body of weight Q on a plane of twice the inclination.*

(14) *Two particles of equal mass, each attracting with a force varying directly as the distance, are situated at the opposite extremities of a diameter of a horizontal circular wire on which a small smooth ring is capable of sliding. Show that the ring will be kept at rest in any position under the attraction of the particles.*

(15) *A body whose weight is W is sustained on a smooth inclined plane by three forces applied to it, each equal to $\frac{W}{3}$. One acts vertically, another horizontally, and the third along the plane. Find the inclination of the plane.*

Ans. Let α be the inclination of the plane. We have, placing the algebraic sum of the components along the plane equal to zero, the condition of equilibrium



$$\frac{W}{3} + \frac{W}{3}\cos\alpha + \frac{W}{3}\sin\alpha - W\sin\alpha = 0.$$

Hence,

$$2\sin\alpha = 1 + \cos\alpha.$$

Or, since $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$ and $1 + \cos\alpha = 2\cos^2\frac{1}{2}\alpha$,

$$2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = \cos^2\frac{\alpha}{2}.$$

Solving this equation, we have

$$\cos\frac{\alpha}{2} = \sin\frac{\alpha}{2} \pm \sin\frac{\alpha}{2},$$

or

$$\cos\frac{\alpha}{2} = 2\sin\frac{\alpha}{2} \text{ or } 0.$$

We have then two values for α, given by $\tan\frac{\alpha}{2} = \frac{1}{2}$, or $\alpha = 53° \; 7' \; 48''.4$ and $\alpha = 180°$.

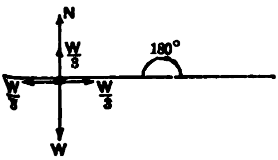
Placing the algebraic sum of the components perpendicular to the plane equal to zero, we have

$$N + \frac{W}{3}\cos\alpha - \frac{W}{3}\sin\alpha - W\cos\alpha = 0.$$

Hence

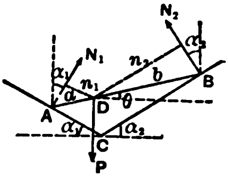
$$N = \frac{W}{3}\left(\sin\alpha + 2\cos\alpha\right).$$

The first value of $\alpha = 53° \; 7' \; 48''.4$ gives $N = +\frac{2}{3}W$. The second value of $\alpha = 180°$ gives $N = -\frac{2}{3}W$. The first value gives a rational solution. The second value corresponds to the case of the particle placed underneath the plane, the normal reaction of the plane being directed towards the plane. If the normal reaction could consist of a *pull*, this position would be possible.



(16) *A rod AB rests on two smooth planes AC and BC which make the angles α_1 and α_2 with the horizontal. A load of P lbs. is applied at a point D of the rod at a distance $AD = a$ and $BD = b$*

from the ends. Find the inclination of the rod to the horizontal when equilibrium exists, and the pressures N_1 and N_2 on the planes. Weight of the rod neglected. (For friction see Ex. (15), page 221.)



Ans. The forces acting upon the rod are the vertical weight P at D and the normal pressures N_1 and N_2 at A and B. These pressures make the same angles with the vertical that the planes AC and BC make with the horizontal.

We have then for equilibrium the algebraic sum of the vertical components equal to zero, or

$$N_1 \cos \alpha_1 + N_2 \cos \alpha_2 - P = 0; \quad \ldots \ldots \quad (1)$$

the algebraic sum of the horizontal components equal to zero, or

$$N_1 \sin \alpha_1 - N_2 \sin \alpha_2 = 0; \quad \ldots \ldots \quad (2)$$

the algebraic sum of the moments about any point in the plane equal to zero. Take the point D and let the lever-arms be n_1 and n_2. Then

$$N_2 n_2 - N_1 n_1 = 0. \quad \ldots \ldots \quad (3)$$

We have from the figure, since n_2 and n_1 are parallel to BC and AC, if θ is the angle of the rod with the horizontal,

$$n_2 = b \cos(\alpha_2 - \theta), \quad n_1 = a \cos(\alpha_1 + \theta),$$

and from (2) we have $N_1 = \dfrac{\sin \alpha_2}{\sin \alpha_1} N_2$. Substituting in (3), we have

$$b \cos(\alpha_2 - \theta) = a \frac{\sin \alpha_2}{\sin \alpha_1} \cos(\alpha_1 + \theta);$$

expanding and reducing, we obtain

$$\tan \theta = \frac{a \operatorname{cotg} \alpha_1 - b \operatorname{cotg.} \alpha_2}{a + b}. \quad \ldots \ldots \quad (4)$$

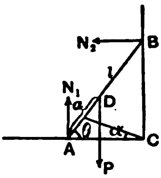
Also from (1) we obtain

$$N_1 = \frac{P \sin \alpha_2}{\sin(\alpha_1 + \alpha_2)}, \quad N_2 = \frac{P \sin \alpha_1}{\sin(\alpha_1 + \alpha_2)}. \quad \ldots \ldots \quad (5)$$

If $\alpha = 90°$ and $\alpha_1 = 0$, or the plane BC is vertical and AC horizontal, we have from (4), $\theta = 90°$, and from (5), $N_1 = P$ and $N_2 = 0$. That is, the position of equilibrium is when the rod is vertical and the end A is at C. If it has any other position, there is no equilibrium unless another force is introduced.

(17) *A rod AB of length l rests upon two smooth planes, one AC horizontal and the other BC vertical, and its inclination with the horizontal is θ. A load of P lbs. is applied at a distance $AD = a$ from the end A. The rod is prevented from sliding by a string attached to C and the rod. If the inclination of this string with the horizontal is α, find the stress in it for equilibrium. Weight of the rod neglected.*

Ans. The forces acting upon the rod are the vertical weight P acting at D, the stress S in the string, and the normal pressures N_1 and N_2 at A and B.



We have then for equilibrium the algebraic sum of the vertical components equal to zero, or

$$N_1 - P - S \sin \alpha = 0; \quad \ldots \ldots \quad (1)$$

the algebraic sum of the horizontal forces equal to zero, or

$$S \cos \phi - N_2 = 0; \quad \ldots \ldots \quad (2)$$

the algebraic sum of the moments about any point in the plane equal to zero.

Take the point C as the centre of moments. Then the lever-arm for N_1 is $l \cos\theta$, for N_2 it is $l \sin\theta$, and for P, $(l-a)\cos\theta$. Hence

$$N_2 l \sin\theta + P(l-a)\cos\theta - N_1 l \cos\theta = 0. \quad . \quad . \quad . \quad . \quad (3)$$

From these three equations we obtain

$$S = \frac{Pa}{l\cos\alpha(\tan\theta - \tan\alpha)}; \qquad N_1 = P + \frac{Pa\tan\alpha}{l(\tan\theta - \tan\alpha)};$$

$$N_2 = \frac{Pa}{l(\tan\theta - \tan\alpha)}.$$

(18) *A body is sustained on a smooth inclined plane of inclination α with the horizon by a force P acting along the plane and a horizontal force H. When the inclination is half α, the forces are $\frac{P}{2}$ and $\frac{H}{2}$, and the body is still at rest. Find the ratio of P to H.*

Ans. $\frac{P}{H} = 2\cos^2\frac{\alpha}{4}$.

(19) *A weight of 10 kilograms is sustained on a smooth inclined plane of 25° inclination with the horizon, by a horizontal force of 5 kilograms and a force unknown in magnitude and direction. Find this force when the normal pressure on the plane is 2 kilograms.*

Ans. 9.07 kilograms making an angle β below the plane of about 88° 6'.

(20) *Find the inclination of a smooth inclined plane if a weight of 24 kilograms resting upon it is sustained by a horizontal force of 7 kilograms and a force of 16 kilograms of unknown direction, while the normal pressure is a force of 15 kilograms. Find also the unknown direction.*

Ans. $\alpha = 53°\ 53'$; $\beta = 17°\ 28'$.

(21) *Find the inclination of a smooth inclined plane if a weight of 20 kilograms resting on it is sustained by force up the plane of 5 kilograms and a force of 15 kilograms of unknown direction, while the normal pressure is 2 kilograms. Find also the unknown direction.*

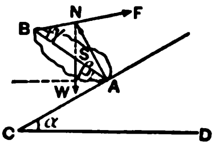
Ans. $\alpha = 49°\ 28'$; $\beta = 47°\ 9'$.

(22) *Find the inclination α of a smooth inclined plane if a given weight W resting on it is sustained by a horizontal force H and a force P of unknown direction, while the normal pressure is N. Find also the unknown direction.*

Ans. For convenience of notation let $A = \frac{W^2 + H^2 + N^2 - P^2}{2N}$. Then

$$\cos\alpha = \frac{AW}{W^2+H^2} \pm \frac{H}{W^2+H^2}\sqrt{W^2+H^2-A^2}, \quad \sin\beta = \frac{W^2+H^2-N^2-P^2}{2PN}.$$

(23) *A rigid body rests at the point A upon a smooth inclined plane ACD which makes an angle α with the horizontal. The axis AB of the body makes an angle β with the horizontal. At the point B a force P is applied which makes an angle γ with the axis AB. At the point s of the body a vertical force W is applied. All the forces act in the plane of AB and AC. Find the conditions of equilibrium.*



Ans. Let $AB = a$, $AS = b$, and the normal pressure at A be N.

The forces acting upon the body are P, W and the normal pressure at A. If these forces are in equilibrium, we have for the algebraic sum of the moments about A

$$Wb\cos\beta - Pa\sin\gamma = 0, \quad \text{or} \quad P = \frac{Wb\cos\beta}{a\sin\gamma}. \quad . \quad . \quad . \quad (1)$$

Placing the algebraic sum of the horizontal components zero, we have

$$P\cos(\gamma-\beta) - N\sin\alpha = 0, \quad \text{or} \quad N = \frac{P\cos(\gamma-\beta)}{\sin\alpha} = \frac{Wb\cos\beta\cos(\gamma-\beta)}{a\sin\gamma\sin\alpha}. \quad (2)$$

If we take moments about B, we have

$$Na\sin(90-\alpha-\beta) - W(a-b)\cos\beta = 0, \quad \text{or} \quad \cos(\alpha+\beta) = \frac{(a-b)\sin\gamma\sin\alpha}{b\cos(\gamma-\beta)}. \quad (3)$$

We thus determine P, N and the direction of the axis AB.

We also have the algebraic sum of the components along the plane equal to zero, or

$$P\cos(\alpha+\beta-\gamma) - W\sin\alpha = 0.$$

Reducing and inserting the values of P and $\cos(\alpha+\beta)$ from (1) and (3), we have

$$\tan(\alpha+\beta) = \frac{a\sin\gamma\sin\beta + b\cos\gamma\cos\beta}{(a-b)\sin\gamma\cos\beta}.$$

Also, since P, N and W must make a closed triangle,

$$N = \frac{W}{a\sin\gamma}\sqrt{(b\cos\beta)^2 + (a\sin\gamma)^2 - 2ab\cos\beta\sin\gamma\sin(\gamma-\beta)}.$$

If P is horizontal, we have $\gamma = \beta$, and

$$P = \frac{Wb}{a}\cot\beta;$$

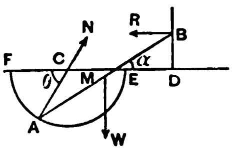
$$N = \frac{Wb}{a}\cdot\frac{\cot\beta}{\sin\alpha}, \quad \text{or} \quad N = \frac{W}{a\sin\alpha}\sqrt{(b\cos\beta)^2 + (a\sin\beta)^2};$$

$$\cos(\alpha+\beta) = \frac{(a-b)}{b}\sin\beta\sin\alpha, \quad \tan(\alpha+\beta) = \frac{a\sin^2\beta + b\cos^2\beta}{(a-b)\sin\beta\cos\beta}.$$

The student should solve by the principle of virtual work.

(24) *The upper end of a rod rests against a smooth vertical plane, and the lower end in a smooth spherical bowl. A weight W acts at any point M of the rod. Find the position of equilibrium.* (For rough surfaces see Ex. (24), page 227.)

Ans. Let AB be the rod, DB the vertical plane and FAE the spherical surface. The forces acting upon the rod are the weight W acting at the point M of the rod, the normal pressure N on the spherical surface which passes through the centre C of the sphere, and the normal pressure R on the vertical plane.



Let α be the angle of the rod with the horizontal and θ the angle of the radius $AC = r$ with the horizontal.

Then we have for equilibrium

$$\left.\begin{array}{l} N\cos\theta - R = 0, \\ N\sin\theta - W = 0, \end{array}\right\} \quad \text{or} \quad N = \frac{W}{\sin\theta}, \quad R = W\cot\theta. \quad . \quad . \quad . \quad (1)$$

Take moments about M. Let the distance $AM = a$ and $MB = b$. Then the lever-arm of R is $b\sin\alpha$, and the lever-arm of N is $a\sin(\theta-\alpha)$, and we have

$$Rb\sin\alpha - Na\sin(\theta-\alpha) = 0,$$

or, substituting the values from (1),

$$a \sin(\theta - \alpha) = b \cos\theta \sin\alpha.$$

Developing and reducing, this becomes

$$(a + b)\tan\alpha = a\tan\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let the length of the rod be l. Then the distance $CD = d$ of the centre of the spherical surface from D is

$$d = l\cos\alpha - r\cos\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

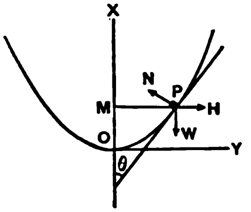
From (2) and (3) we can determine α and θ. The position of equilibrium is independent of W, but depends upon the position of W and C.

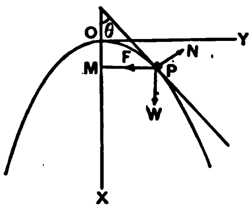
(25) *A body whose weight is W is at rest upon a smooth parabolic curve whose axis is vertical, and is acted upon at any point P by a horizontal force H whose magnitude is always proportional to the distance PM from the axis. Find the position for equilibrium.* (For rough surface see Ex. (25), page 227.)

Ans. The equation of the parabola, taking the origin at the vertex O, is

$$y^2 = 2px,$$

where the axis of X is vertical and the axis of Y horizontal and p is the ordinate to the curve through the focus.





We consider the body, whatever its size as a particle, acted upon by concurring forces (page 169). The applied forces are W, H and the normal reaction of the curve. These make a system of concurring forces in equilibrium.

Let the horizontal force which acts upon the particle when it is at the distance p from the axis be H_1. Then the force H when it is at any other distance $PM = y$ from the axis is

$$H = \frac{y}{p} H_1.$$

Let θ = angle between the tangent at P and the vertical.

Then, taking the algebraic sum of all the components along the tangent, we have for equilibrium the condition

$$W\cos\theta - H\sin\theta = 0.$$

This condition holds whether the particle rests within the curve or upon it. Substitute the value of H, and we have for the condition of equilibrium

$$W\cos\theta = \frac{y}{p} H_1 \sin\theta.$$

This condition is evidently satisfied when $\theta = 90°$ and $y = 0$, that is, when the particle is at the vertex.

If the particle is not at the vertex, we have

$$\tan\theta = \frac{Wp}{H_1 y}.$$

But if the curve is a parabola, we have for any point $\tan\theta = \frac{p}{y}$. Hence the condition for equilibrium for any point is $H_1 = W$.

If then the magnitude of the horizontal force when the particle is at the distance p from the axis is W, the particle will be at rest at any point of the curve. If it is not, the vertex is the only position.

(26) *A body of weight W, resting on a smooth inclined plane, is attached to a string which, passing over a smooth pulley, sustains a body of weight P. If β is the inclination of the string to the inclined plane and α the inclination of the plane to the horizon, find the conditions and position of equilibrium.*

Ans. (Example (1).) The condition of equilibrium is $P\cos\beta = W\sin\alpha$, or $\cos\beta = \frac{W\sin\alpha}{P}$.

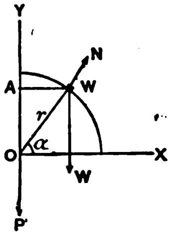
Since β must be less than 90°, $\cos\beta$ must be less than unity. Hence $W\sin\alpha$ must be less than P. If the condition of equilibrium is satisfied for one point of the plane, it will be satisfied for all others.

(27) *A body whose weight is 10 kilograms is supported on a smooth inclined plane by a force of 2 kilograms acting along the plane and a horizontal force of 5 kilograms. Find the inclination of the plane and the normal reaction.*

Ans. $\alpha = 36°\ 52'\ 11''$, $\sin\alpha = \frac{3}{5}$, $\cos\alpha = \frac{4}{5}$; $N = 11$ kilograms.

(28) *Two weights P and W are fastened to the ends of a cord which passes over a smooth pulley O. The weight W rests upon a smooth vertical plane curve and P hangs freely. Find the position of equilibrium (a) when the curve is a parabola and O is at the focus; (b) when the curve is a circle and O is at a distance a above the centre; (c) when the curve is an hyperbola and O is at the centre, the axis of the curve being vertical; (d) find the curve such that the weight W may be in equilibrium with P at all points of the curve.*

Ans. The applied forces are the weight W acting vertically, the tension P of the string and the normal reaction N of the curve.



Take the origin at O and let OW make the angle α with the horizontal. Then, since $AW = x$, $OA = y$, if we denote OW by r, we have

$$\sin\alpha = \frac{y}{r},\quad \cos\alpha = \frac{x}{r},\quad r^2 = x^2 + y^2.$$

Let N make the angle θ with the vertical, then the tangent at W makes the same angle with the horizontal.

We have then for the algebraic sum of the vertical components

$$N\cos\theta - W - P\sin\alpha = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and for the algebraic sum of the horizontal components

$$N\sin\theta - P\cos\alpha = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2) we obtain

$$\tan\theta = \frac{P\cos\alpha}{W + P\sin\alpha}.$$

The tangent of the angle which the tangent to the curve at W makes with the horizontal is then for equilibrium

$$\frac{dy}{dx} = -\tan\theta = -\frac{P\cos\alpha}{W + P\sin\alpha} = -\frac{Px}{Wr + Py}. \quad . \quad . \quad . \quad (3)$$

Equation (3) is general whatever the curve. We may obtain it directly from equation (6), page 172. Thus $F_x = -P\cos\alpha = -\frac{Px}{r}$, $F_y = -W - P\sin\alpha = -W - \frac{Py}{r}$. Hence, since $F_x dx + F_y dy = 0$, we have at once equation (3).

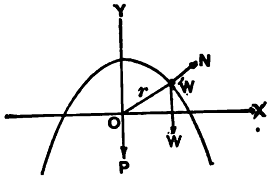
(*a*) If the curve is a parabola with origin at the focus O and axis vertical, the equation of the curve, since y is negative downwards, is

$$x^2 = -2py + p^2, \quad \text{or} \quad u = x^2 + 2py - p^2 = 0,$$

where p is twice the distance from the focus to the vertex.

Differentiating, we have

$$\frac{dy}{dx} = -\tan\theta = -\frac{x}{p}.$$



Substituting in (3), we obtain for equilibrium

$$W = P\frac{p-y}{r} = P\frac{p-y}{\pm\sqrt{x^2+y^2}}$$

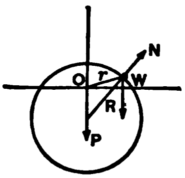
$$= P\frac{p-y}{\pm\sqrt{y^2-2py+p^2}} = \pm P.$$

Hence equilibrium obtains when W and P are equal and holds good for any point on the curve. We may obtain the same result directly from equation (5), page 172. Thus

$$\frac{du}{dx} = 2x, \quad \frac{du}{dy} = 2p, \quad F_x = -\frac{Px}{r}, \quad F_y = -W - \frac{Py}{r}.$$

Substituting in $\dfrac{F_x}{\left(\dfrac{du}{dx}\right)} = \dfrac{F_y}{\left(\dfrac{du}{dy}\right)}$, we obtain at once $W = P\dfrac{p-y}{r} = \pm P$.

(*b*) If the curve is a circle with the origin and pulley at a distance a above the centre of the circle, the equation of the circle, since y is negative downwards, is



$$(a+y)^2 + x^2 = R^2, \quad \text{or} \quad u = R^2 - x^2 - (a+y)^2 = 0,$$

where R is the radius.

Differentiating, we have

$$\frac{dy}{dx} = -\tan\theta = -\frac{x}{a+y}.$$

Substituting in (3), we obtain for equilibrium

$$r = \frac{P}{W}a.$$

We may obtain the same result directly from equation (5), page 172, by inserting the values

$$\frac{du}{dx} = -2x, \quad \frac{du}{dy} = -2a - 2y, \quad F_x = -\frac{Px}{r}, \quad F_y = -W - \frac{Py}{r}.$$

(*c*) If the curve is an hyperbola with the origin and pulley at the centre of the hyperbola, the axis of the curve being vertical, the equation of the curve is

$$b^2y^2 - a^2x^2 = a^2b^2, \quad \text{or} \quad h = b^2y^2 - a^2x^2 - a^2b^2 = 0.$$

Differentiating, we have

$$\frac{dy}{dx} = -\tan\theta = \frac{a^2x}{b^2y}.$$

Substituting in (3), we obtain for equilibrium

$$y = \frac{bW}{e\sqrt{W^2 - e^2P^2}}$$

where e is the eccentricity or $e = \sqrt{\frac{a^2+b^2}{a^2}}$.

We may obtain the same result from equation (5), page 172, by substituting

$$\frac{du}{dx} = -2a^2x, \quad \frac{du}{dy} = 2b^2y, \quad F_x = -\frac{Px}{r}, \quad F_y = -W - \frac{Py}{r}, \quad r^2 = x^2 + y^2.$$

(*d*) Required the curve such that the weight W may be in equilibrium with the weight P for all points of the curve.

We have from (3)

$$\frac{dy}{dx} = -\frac{Px}{Wr + Py} = -\frac{Px}{W\sqrt{x^2+y^2} + Py},$$

or

$$-Wdy = P\frac{xdx + ydy}{\sqrt{x^2+y^2}}.$$

Integrating, we have

$$-Wy + C = P\sqrt{x^2+y^2}.$$

Squaring,

$$W^2y^2 - 2CWy + C^2 = P^2x^2 + P^2y^2.$$

Hence

$$P^2x^2 + (P^2 - W^2)y^2 + 2CWy - C^2 = 0.$$

This is an equation of the second degree and is therefore a conic section.

If $P = W$, it is a parabola;
$P > W$, it is an ellipse;
$P < W$, it is an hyperbola;
the origin and pulley being at the focus.

(29) *A particle whose weight is W is placed on the concave surface of a smooth sphere and is acted upon by gravity and also by a repulsive force varying inversely as the square of the distance from the lowest point of the sphere. Find the position of equilibrium.**

Ans. Take the lowest point of the sphere as the origin, and let the axis of Y be vertical.

The equation of the surface is, if R is the radius,

$$u = x^2 + y^2 + z^2 - 2Ry = 0.$$

Let r be the distance of the particle from the lowest point of the sphere. Then

$$r^2 = x^2 + y^2 + z^2 = 2Ry. \quad \ldots \ldots \quad (a)$$

Let the repulsive force at a known distance a from the lowest point be F_1. Then the repulsive force at any distance r will be $F_1\frac{a^2}{r^2} = F_1\frac{a^2}{2Ry}$.

Let the repulsive force make the angles α, β, γ with the co-ordinate axes. Then $\cos\alpha = \frac{x}{r}$, $\cos\beta = \frac{y}{r}$, $\cos\gamma = \frac{z}{r}$, and the component forces parallel to the co-ordinate axes are

$$F_x = F_1\frac{a^2}{2Ry}\cdot\frac{x}{r}, \quad F_y = F_1\frac{a^2}{2Ry}\cdot\frac{y}{r} - W, \quad F_z = F_1\frac{a^2}{2Ry}\cdot\frac{z}{r}.$$

* This is the problem of the electroscope.

Hence from equation (3), page 172, we have after reduction

$$y = \frac{1}{2}\sqrt[3]{\frac{a^4F_1^2}{RW^2}}.$$

Inserting this in (a), we obtain

$$r^3 = \frac{a^2F_1R}{W}.$$

If another force of the same kind makes the particle rest at a distance r' from the lowest point, and if F_1' is the force at a distance a', then

$$r'^3 = \frac{a'^2F_1'R}{W},$$

and hence

$$\frac{r^3}{r'^3} = \frac{a^2F_1}{a'^2F_1'};$$

that is, the values of the repulsive forces at distance unity vary as the cubes of the distance from the lowest point.

Substituting the values of F_x, F_y, F_z in equation (4), page 172, and the values of y and r already found, we obtain

$$xdx + ydy + zdz - Rdy = 0,$$

which is the differential equation of equation (a).

CHAPTER IX.

CONSTRAINED EQUILIBRIUM—ROUGH CURVE OR SURFACE.

FRICTION. ADHESION. KINDS OF FRICTION. REACTION OF A ROUGH CURVE OR SURFACE. EQUILIBRIUM OF A BODY ON A ROUGH CURVE OR SURFACE. ANGLE OF FRICTION OR REPOSE. CONE OF FRICTION. COEFFICIENT OF FRICTION. LIMITING EQUILIBRIUM. COEFFICIENT OF STATIC SLIDING FRICTION. LAWS OF STATIC SLIDING FRICTION. VALUES OF COEFFICIENT OF STATIC SLIDING FRICTION. STATIC FRICTION OF PIVOTS. STATIC FRICTION OF AXLES. STATIC FRICTION OF CORDS AND CHAINS. RIGIDITY OF ROPES. STATIC ROLLING FRICTION. EQUILIBRIUM OF A BODY AT ANY POINT OF A ROUGH CURVE OR SURFACE. GENERAL EQUATIONS. STABLE, UNSTABLE, INDIFFERENT AND NEUTRAL EQUILIBRIUM. CRITERION FOR STABLE, UNSTABLE, INDIFFERENT AND NEUTRAL EQUILIBRIUM. STABILITY IN ROLLING CONTACT.

Friction. — In the preceding Chapter we have considered the equilibrium of a body on a smooth curve or surface, that is, a curve or surface incapable of offering resistance to motion in any other than a normal direction.

But every natural surface offers a resistance to the motion of a body upon it. Part of this resistance is due to **adhesion** between the body and surface and part is due to **friction**.

Friction then is always a retarding force or resistance, and acts always in a direction opposite to that in which the body moves or would move if there were no resistance.

When one surface moves upon another, the surfaces in contact are compressed and projecting points and irregularities are bent over, broken off, rubbed down, etc.

The resistance due to friction, therefore, evidently depends upon the materials of which the surfaces are composed, and also upon the roughness or smoothness of the surfaces in contact.

It may also evidently vary for the same surfaces, according to their condition or state or material constitution.

Thus it may not be the same for surfaces of dry wood or iron as for the same surfaces under the same conditions when wet. It may not be the same for two surfaces of wood with their fibres parallel as for the same surfaces under the same conditions when their fibres are not parallel.

Unguents also have a great influence. Such fluid or semi-fluid unguents as oil, tallow, etc., fill up interstices and diminish the effect of irregularities of surfaces ; or a film of unguent may be interposed between the surfaces and thus the resistance of friction greatly diminished.

Adhesion.—We must not confound the resistance due to friction with that due to adhesion. Adhesion is that resistance to motion which takes place when two different surfaces come in contact at many points without pressure. Adhesion increases with the area of surface of contact and is independent of the pressure, while, as we shall see (page 191), friction increases with the pressure and is in general independent of the area of surface of contact. When the pressure then is very small, adhesion may be great compared with friction.

If, however, the pressure is great, adhesion may be neglected compared to the friction, and the resistance to motion is practically that due to the friction only.

When the surfaces in contact are of the same kind, we call the resistance to motion **cohesion**; when of different kinds, **adhesion**.

Kinds of Friction.—Surfaces may slide or roll on one another. We distinguish accordingly **sliding friction** and **rolling friction.**

It is also found by experiment that the friction which just prevents motion is greater than that which exists after actual motion takes place. The friction which just prevents motion is called friction of repose or quiescence, or **static friction**. The friction which exists after actual motion takes places is called friction of motion, or **kinetic friction**.

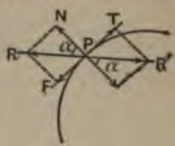
We have then two kinds of static friction, viz., **static sliding friction** and **static rolling friction.**

We have also two kinds of kinetic friction, viz., **kinetic sliding friction** and **kinetic rolling friction.**

In any case, whether of sliding or rolling, the kinetic friction is always less than the static friction.

We have to do in this portion of our work with static friction only.

Reaction of a Rough Curve or Surface.—We have already defined (page 169) the **reaction** of a curve or surface as the pressure which the curve or surface exerts upon a particle in contact with it.



Suppose then a particle in equilibrium at any point P of a rough curve or surface. Let R be the reaction of the curve or surface, and R' the resultant of all other forces acting upon the particle.

Then for equilibrium R and R' must be equal and opposite and make the same angle α with the normal to the curve or surface at the point P.

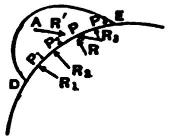
Now R' can be resolved into a normal component which must be resisted by the normal pressure N of the curve or surface at the point P, and into a tangential component T which tends to cause *sliding* and must be resisted by the friction F. The components of the reaction R are then N and F, and we have for equilibrium

$$R \cos \alpha = N, \quad R \sin \alpha = F,$$

$$\tan \alpha = \frac{F}{N}.$$

Hence, *when a particle is in equilibrium at any point of a rough curve or surface, the reaction makes with the normal at this point an angle whose tangent is given by the ratio of the friction to the normal pressure at the point. If the reaction is normal, there is no friction.*

Equilibrium of a Body on a Rough Curve or Surface. — We have seen, page 169, that a body in equilibrium upon any surface, rough or smooth, may be treated as a particle placed at any one of the points of contact with the curve or surface. Also, if the curve or surface exerts pressure only, the resultant R' of all the external forces must intersect the curve or surface at some point P within the line or surface of contact.



We have also just proved that when a particle is in equilibrium at any point of a rough curve or surface, the reaction R makes with the normal at this point an angle α whose tangent is given by the ratio of the friction to the normal pressure.

If then the body ADE rests in equilibrium upon a rough curve or surface and touches it at many points P_1, P_2, P_3, etc., each of the reactions R_1, R_2, R_3, etc., at each of these points makes with the normal at its point an angle α_1, α_2, α_3, etc., whose tangent is given by the ratio $\frac{F_1}{N_1}$, $\frac{F_2}{N_2}$, $\frac{F_3}{N_3}$, etc., of the friction to the normal pressure at each point.

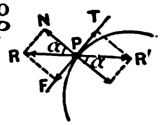
The entire body can then be treated as a particle at any one of the points of contact. The point P where the line of direction of the resultant R' of all the external forces intersects the curve or surface, if the curve or surface exerts pressure only, must lie inside the line or surface of contact DE.

The resultant reaction R at any point of contact of all the forces acting upon the body *except the reaction at this point*, must make with the normal at this point an angle α whose tangent is given by the ratio of the total friction to the resultant normal pressure.

Angle of Friction or Repose.—Let a body be in equilibrium at any point P of a rough curve or surface.

Let R be the reaction of the curve or surface at the point P, and let R' be the resultant of all the external forces acting upon the body.

Then for equilibrium, R is equal and opposite to R' and makes the same angle α with the normal at P given by



$$\tan \alpha = \frac{F}{N}$$

where F is the friction at the point P, and N is the normal pressure at this point.

Now the force which tends to cause sliding is the tangential component of R' or $T = R' \sin \alpha$. The friction F at P acts opposite to T, and so long as there is equilibrium is equal to it.

As the angle α increases, the normal pressure $N = R \cos \alpha$ decreases and the tangential force $T = R' \sin \alpha$ increases. There is evidently a certain value for α for which, R' remaining unchanged in magnitude, *sliding is just about to begin.* For any value of α less than this, sliding cannot begin no matter what the magnitude of R'. For any value of α greater than this, sliding takes place.

We denote this value of α by ϕ and call it the **angle of friction or repose.**

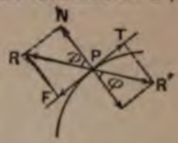
We have then

$$\tan \phi = \frac{\text{max. } F}{\text{min. } N}.$$

That is, the angle of friction or repose is the greatest angle which the reaction R at any point of contact can make with the normal at that point without sliding taking place. Since static friction is always greater than kinetic, it is also the greatest angle which the reaction R at any point of contact can *ever make* with the normal at that point. It is also the greatest angle which the resultant R' of all the external forces acting upon the body can make with the normal at the point without sliding taking place. No resultant force R', however great, can cause sliding to begin, so long as its angle α with the normal is less than the angle of friction or repose.

Cone of Friction.—If then the reaction R at any point of contact P makes the angle of friction or repose ϕ with the normal at that point, *sliding is about to begin.*

If we revolve the line representative of R about the normal at P, it describes the surface of a cone every element of which makes the angle of repose ϕ with the normal. This cone is called the **cone of friction.**



No force acting at the point P, however great in magnitude, can cause sliding to begin at that point if its line representative lies within the cone. The cone of friction encloses the direction of all forces which are completely counteracted by the surface at any point.

Coefficient of Friction.—When two surfaces are in contact and there is friction and normal pressure at every point of contact, the sum of the frictions at every point of contact is the total friction, and the sum of the normal pressures at every point of contact is the total normal pressure.

The ratio of the total friction to the total normal pressure when motion, either sliding or rolling, is *just about to begin*, is called the **coefficient of static friction**, either of sliding or rolling.

The same ratio *after motion has taken place* is called the **coefficient of kinetic friction**, either of sliding or rolling.

We denote the coefficient of friction in general by μ. We have then, in general, for all cases

$$\mu = \frac{F}{N}, \quad \text{or} \quad F = \mu N,$$

where F is the total friction and N the total normal pressure, when motion either sliding or rolling is *just about to begin*, or else when motion either sliding or rolling *has taken place.* In the first case μ is the coefficient of static friction of sliding or rolling. In the second case μ is the coefficient of kinetic friction of sliding or rolling. We have to do in this portion of the work with static friction only.

Limiting Equilibrium.—The student should carefully note that

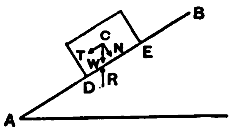
$$F = \mu N$$

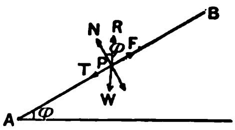
does not give the actual resistance of friction in all cases of equilibrium, but only the resistance which exists when the surfaces *are on the point of motion.*

Friction acts always in a direction opposite to the force which tends to cause motion, and so long as there is equilibrium it is always equal in magnitude to this force. But when this force has the magnitude μN motion is *just about to begin*, and the body is

said to be in **limiting equilibrium.** If this force is less than μN, there will still be equilibrium, whatever its magnitude, and the body is in **non-limiting equilibrium.**

Coefficient of Static Sliding Friction—Experimental Determination.—Let a body of weight W, acting at the centre of mass C, rest in equilibrium upon a rough plane AB, the surfaces of contact being plane.





Then for equilibrium the line of direction of W must intersect the plane inside the base or surface of contact DE, and we can consider the body as a particle placed at the point where W intersects the base, and in equilibrium under the action of the reaction *at that point* and the weight W.

Then the sum N of all the normal pressures acting at every point of contact must be equal and opposite to the normal component of W, and the sum F of all the frictions at every point of contact must be equal and opposite to the component T of W parallel to the plane.

We have then when sliding is about to begin, for the coefficient of sliding friction,

$$\mu = \frac{F}{N},$$

and we see from the figure that $\frac{F}{N}$ is the tangent of the angle which the total reaction R makes with the normal when sliding is about to begin. Now the reaction at every point of contact is parallel to R or W and sliding begins at all points of contact simultaneously. Hence the angle which R makes with the normal when sliding is about to begin is the angle of repose ϕ, and it is evidently the same as the angle which the plane makes with the horizontal. Therefore

$$\mu = \frac{F}{N} = \tan \phi.$$

That is, *the coefficient of static sliding friction is equal to the tangent of the angle of repose.*

If, then, we place a body upon a rough plane and then gradually incline the plane until sliding just begins, the inclination of the plane at this instant gives the angle of friction or repose ϕ. The tangent of this angle gives the coefficient μ of static sliding friction for plane surfaces.

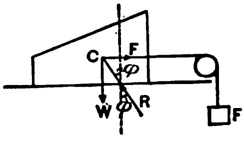
We obtain the same result by resolution of forces. Thus let ϕ be the inclination of the plane when sliding begins.

Then for equilibrium $W \cos \phi = N$, and $W \sin \phi = F$. Hence

$$\mu = \frac{F}{N} = \tan \phi.$$

We can thus make use of the inclined plane as an apparatus for determining μ by experiment.

Again, if we place a body of weight W on a horizontal plane and measure the horizontal force F just necessary to cause it to begin to slide, we have



$$\mu = \frac{F}{W} = \tan \phi,$$

where ϕ is the angle of the reaction R with the normal when sliding begins, or the angle of repose.

Such an apparatus should be so constructed that the friction of the pulley and other resistances due to the string, etc., can be disregarded or else allowed for.

Laws of Static Sliding Friction.—The following laws of static sliding friction have been established by experiment as holding true within the limits indicated :

1. *Other things being the same, within certain limits of the normal pressure, static sliding friction is proportional to the total normal pressure and independent of the area of the surfaces in contact.*

In other words, within the limits of normal pressure referred to, the coefficient of static sliding friction μ *is constant for the same two surfaces in the same condition*, whatever the area of the surfaces of contact and whatever the total normal pressure.

Thus, if the normal pressure N over a given area is increased or decreased, the friction F increases or decreases in the same proportion and $\mu = \frac{F}{N}$ is unchanged.

It follows directly that if the area increases or decreases, N remaining the same, the number of points of contact is correspondingly increased or decreased, but the normal pressure at each point, and therefore the friction at each point, is correspondingly decreased or increased. The sum of all the frictions F remains then the same and $\mu = \frac{F}{N}$ is unchanged.

Limitations of the Law.—The limitations of normal pressure referred to are as follows:

If the normal pressure per unit of area approaches the crushing strength or becomes so great as to break up the film of interposing unguent, the friction F increases more rapidly than the normal pressure and the law fails.

In properly designed structures the normal pressure per unit of area is much less than this limit and the law applies.

Again, if the normal pressure per unit of area is very small, adhesion may constitute the larger portion of the resistance. This adhesion increases with the area of contact (page 187).

In all practical cases, however, the influence of adhesion may be neglected.

Hence in practical applications the friction is the only resistance which is considered, and it is assumed that

$$F = \mu N$$

gives the resistance, where μ is in practice a constant for the same two surfaces in the same condition, whatever the area of the surfaces in contact and whatever the total normal pressure N.

2. *Other things being the same, within certain limits of the normal pressure, the stating sliding friction of greased surfaces is less than that of ungreased and depends less upon the surfaces than upon the unguent.*

Here again, if the normal pressure per unit of area becomes so great as to break up the film of interposing unguent, surface comes in contact with surface and the friction may depend more on the surfaces than upon the unguent.

In properly designed structures the normal pressure per unit of area is much less than this, and the law applies.

Again, if the normal pressure per unit of area is very small, adhesion may constitute the larger portion of the resistance and this adhesion is increased by the unguent.

In all practical cases, however, the influence of adhesion may be neglected.

Hence in practical applications the friction is the only resistance which is considered and it is assumed that

$$F = \mu N$$

gives the resistance, where μ is in practice a constant for the same two surfaces in the same condition, whatever the area of the surfaces in contact and whatever the total normal pressure N.

Upon these two laws depend the value and use of the values for the coefficient of static sliding friction given in the next Article.

Values of Coefficient of Static Sliding Friction.—The following table gives a few values of the value of μ as determined by experiment for static sliding friction.

COEFFICIENTS OF STATIC SLIDING FRICTION $\mu = \tan \phi$.

Substances in Contact.	Condition of Surfaces and Kind of Unguent. Dry.	Wet.	Olive Oil.	Lard.	Tallow.	Dry Soap.	Polished and Greasy.
Wood on wood, minimum	0.30	0.65	....	....	0.14	0.22	0.30
Wood on wood, mean	0.50	0.68	....	0.21	0.19	0.36	0.35
Wood on wood, maximum	0.70	0.71	....	....	0.25	0.44	0.40
Metal on metal, minimum	0.15	....	0.11	....	....	....	....
Metal on metal, mean	0.18	...	0.12	0.10	0.11	....	0.15
Metal on metal, maximum	0.24	....	0.16	....	....	....	....
Wood on metal	0.60	0.65	0.10	0.12	0.12	....	0.10
Hemp ropes or plaits on wood, minimum	0.50	....					
Hemp ropes or plaits on wood, mean	0.63	0.87					
Hemp ropes or plaits on wood, maximum	0.80						
Leather belts over drums made of wood	0.47						
Leather belts over drums made of metal	0.54	....	....	....	....	....	0.28
Stone or brick on stone or brick, polished, minimum	0.67						
Stone or brick on stone or brick, polished, maximum	0.75						
Dry masonry and brickwork	0.65						
Masonry and brickwork, damp mortar	0.74						
Timber on stone	0.40						
Iron on stone	0.7 to 0.8						
Masonry on dry clay	0.51						
" " moist clay	0.33						
Earth on earth	0.25 to 1						
Damp clay on damp clay	1.0						

More extensive tables will be found in treatises on Engineering. It will be noted that the coefficient of static sliding friction is practically always less than unity. In only one case given in the table, viz., for damp clay on damp clay, is $\mu = 1$, corresponding to

an angle of repose of $\phi = 45°$. Rankine gives for "shingle on gravel" a maximum $\mu = 1.11$, corresponding to an angle of repose $\phi = 48°$.

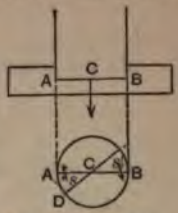
Static Friction for Pivots.—In all cases of the sliding of two surfaces, we denote the coefficient of static sliding friction by μ and take the value of μ as given by the Table page 192. We have then *in all cases* of sliding friction, for the friction *when sliding is about to begin*,

$$F = \mu N = N \tan \phi,$$

where N is the *total normal pressure* and ϕ is the angle of repose, and μ is given by the Table page 192. The direction of the friction is always opposite to the direction of motion if motion were to take place.

The application to pivots is then simple.

1. Solid Flat Pivot.—Let ACB be the base of a solid flat pivot and N the total normal pressure upon the base.



We have then for the static friction

$$F = \mu N, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

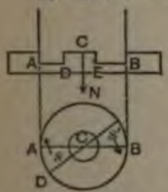
where μ is given by the Table page 192.

If we divide the base into a very large number of very small equal triangles such as ACD, the friction on each can be considered as the resultant of equal parallel forces distributed over the surface. The point of application for each triangle is then at the centre of mass for that triangle. The point of application of the entire friction is then at a distance $Cs = \frac{2}{3}r$ from the centre. The *moment* of the entire friction with reference to the axis is then

$$M = \frac{2}{3}\mu Nr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since for any point s of the base there is a corresponding point s' for which the friction is equal and opposite, the moment of the friction is the moment of a *couple*, and is therefore the same for every point in the plane of the base (page 72).

2. Hollow Flat Pivot.—If the rubbing surface is a flat ring $ADEB$, we have as before



$$F = \mu N, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where N is the total normal pressure on the base and μ is the coefficient of static sliding friction as given by the Table page 192.

Let the outer radius be r_1 and the inner radius r_2. Then any small portion of the base is a circular ring for which the length of chord and arc AD may be taken equal. The centre of mass (page 25) for each small portion is then at a distance Cs from the axis given by

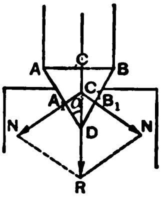
$$Cs = \frac{2}{3}\,\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}.$$

Hence the moment of the friction with reference to the axis is

$$M = \frac{2}{3}\mu N\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since for any point s there is a corresponding point s' for which the friction is equal and opposite, the moment of the friction is the moment of a couple and is therefore the same for any point in the plane of the base (page 72).

3. Conical Pivot.—In the case of a conical pivot let R be the pressure along the axis and let the half angle of convergence ADC be α.



If we divide the conical surface into a large number n of very small triangles with their vertices at the point D, each will sustain the vertical load $\frac{R}{n}$, and the normal pressure on each will be $\frac{R}{n \sin \alpha}$. If we denote the radius $C_1A_1 = C_1B_1$ of the pivot at the point of entrance by r_1, the resultant normal pressure upon each small elementary triangle acts at a distance $\frac{2}{3}r_1$ from the axis.

We have then for the total friction

$$F = \mu \frac{R}{\sin \alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where μ is the coefficient of static sliding friction as given by the Table page 192, and the moment of the friction with reference to the axis is

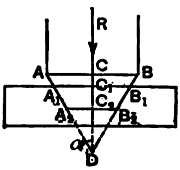
$$M = \frac{2}{3}\mu \frac{Rr_1}{\sin \alpha},$$

or, since $\frac{r_1}{\sin \alpha}$ = the side DA_1 of the cone of contact $= a$, we have

$$M = \frac{2}{3}\mu Ra. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is also the moment of a couple and hence the same for any point in the plane perpendicular to the axis at a distance above the point D equal to two thirds the height of the cone of contact.

4. Pivot a Truncated Cone.—Let R be the pressure along the axis and let the half angle of convergence ADC be α.



Let R_2 be the pressure sustained by the flat base and R_1 the pressure sustained by the conical surf,ace.

Then

$$R_1 + R_2 = R.$$

Also, if r_1 is the radius C_1A_1 at the point of entrance and r_2 the radius of the base,

$$R_2 : R :: \pi r_2^2 : \pi r_1^2, \quad \text{or} \quad R_2 = \frac{r_2^2}{r_1^2}R,$$

and hence

$$R_1 = R - R_2 = \frac{r_1^2 - r_2^2}{r_1^2}R.$$

We have then as in Case 1, page 193, for the flat pivot, the friction F_2 on the base

$$F_2 = \mu R_2 = \mu \frac{r_2^2}{r_1^2}R,$$

and its moment about the axis

$$M_2 = \frac{2}{3}\mu\frac{r_2^3}{r_1^2}R.$$

For the friction on the conical surface we have, as in Case 3, page 194, for the conical pivot

$$F_1 = \mu\frac{R_1}{\sin\alpha} = \mu \cdot \frac{r_1^2 - r_2^2}{r_1^2} \cdot \frac{R}{\sin\alpha},$$

and for its lever-arm, as in Case 2, page 193, for hollow pivot,

$$\frac{2}{3} \cdot \frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}.$$

Its moment then about the axis is

$$M_1 = \frac{2}{3}\mu \cdot \frac{r_1^3 - r_2^3}{r_1^2} \cdot \frac{R}{\sin\alpha}.$$

The total friction for the truncated pivot is then

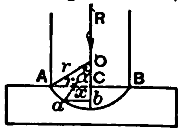
$$F = F_1 + F_2 = \frac{\mu R}{r_1^2}\left(r_2^2 + \frac{r_1^2 - r_2^2}{\sin\alpha}\right), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and its total moment about the axis is

$$M = M_1 + M_2 = \frac{2}{3}\mu\frac{R}{r_1^2}\left(r_2^3 + \frac{r_1^3 - r_2^3}{\sin\alpha}\right), \quad . \quad . \quad . \quad . \quad (2)$$

where μ is the coefficient static sliding friction as given by the Table page 192.

[**Pivot with Spherical End.**]—Let R be the pressure along the axis, denote the radius AO of the spherical surface by r, and the radius AC by r_1, and let the angle AOC be α.



Then the load per unit of area of horizontal projection is $\frac{R}{\pi r_1^2}$. Take any element of the surface at a, distant $ab = x$ from the axis, and let $Ob = y$. The horizontal projection of this element is $2\pi x dx$ and the load sustained by it is then $2\pi x dx \times \frac{R}{\pi r_1^2} = \frac{2Rxdx}{r_1^2}$.

The cosine of the angle aOb is $\cos aOb = \frac{y}{r} = \frac{\sqrt{r^2 - x^2}}{r}$. The normal pressure on the element at a is then

$$\frac{2Rxdx}{r_1^2} \cdot \frac{r}{\sqrt{r^2 - x^2}},$$

and the static friction is

$$\frac{2\mu Rr}{r_1^2} \cdot \frac{xdx}{\sqrt{r^2 - x^2}}.$$

Integrating between the limits of $x = 0$ and $x = r_1$, we have for the total friction

$$F = \frac{2\mu Rr}{r_1^2}\left(r - \sqrt{r^2 - r_1^2}\right),$$

or, since $\sqrt{r^2 - r_1^2} = r\cos\alpha$ and $r_1 = r\sin\alpha$,

$$F = \frac{2\mu R}{\sin^2\alpha}\left(1 - \cos\alpha\right) = \frac{2\mu R}{1 + \cos\alpha},$$

where μ is the coefficient of static sliding friction as given by the Table page 192.

For hemispherical end $\alpha = 90°$ and $F = 2\mu R$. For flat end $\alpha = 0$ and $F = \mu R$.

The moment about the axis of the friction on an element is

$$\frac{2\mu Rr}{r_1^2} \cdot \frac{x^2dx}{\sqrt{r^2 - x^2}}.$$

Integrating between the limits $x = 0$ and $x = r_1$, we have for the total moment of the friction about the axis

$$M = \frac{2\mu Rr}{r_1^2}\left[\frac{r^2}{2}\sin^{-1}\frac{r_1}{r} - \frac{r_1}{2}\sqrt{r^2 - r_1^2}\right],$$

or, inserting the values of $\sqrt{r^2 - r_1^2} = r\cos\alpha$ and $r_1 = r\sin\alpha$ and reducing,

$$M = \mu Rr\left(\frac{\alpha}{\sin^2\alpha} - \cot\alpha\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For hemispherical end $\alpha = \frac{\pi}{2}$, $\sin\alpha = 1$, $\cot\alpha = 0$, and this becomes $M = \frac{\mu\pi Rr}{2}$.

Static Friction of Axles.—In all cases of the sliding of two surfaces, we denote the coefficient of static sliding friction by μ and take the value of μ as given by the Table page 192. We have then *in all cases* of sliding friction for the friction, when ***sliding is about to begin***,

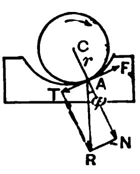
$$F = \mu N = N\tan\phi,$$

where N is the total normal pressure and ϕ is the angle of repose, and μ is given by the Table page 192.

The direction of the friction is always opposite to the direction of motion if motion were about to take place.

The application to axles is then simple.

1. **Axle in Partially Worn Bearing.**—Let the bearing be partially worn, then the axle at the moment when sliding begins touches the bearing at a point A, and the resultant pressure R at this point makes the angle of repose ϕ with the normal. We have then for the normal pressure $N = R\cos\phi$, and for the friction



$$F = N\tan\phi = R\sin\phi, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where ϕ is the angle of repose as given by the Table page 192.

Let r be the radius AC of the axle. Then the moment of the friction with reference to the axis is

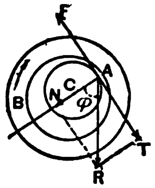
$$M = Rr\sin\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the axle is well greased, the angle of repose ϕ is very small and we may take $\mu = \tan\phi = \sin\phi$. In the practical case of a well-greased axle, then, we have

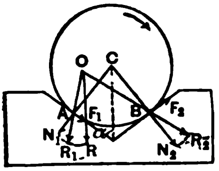
$$F = \mu R, \quad M = \mu Rr,$$

where μ is given by the Table page 192.

If the wheel AB revolves, as shown, about a fixed axle AC, the friction is the same as before, but the lever-arm of the friction is not the radius of the axle, but the inner radius of the wheel.



2. Axle—Triangular Bearing.—If the bearing is triangular, the axle is supported at two points A and B. The resultant pressure R can be resolved into two components R_1 and R_2, and when sliding begins, each of these makes the angle of repose ϕ with the normals at A and B. The normal pressure at A is then $N_1 = R_1 \cos\phi$, and the friction at A is



$$F_1 = N_1 \tan\phi = R_1 \sin\phi.$$

The friction at B is in like manner $F_2 = R_2 \sin\phi$. The total friction is then

$$F = (R_1 + R_2)\sin\phi.$$

Let the angle $ACB = 2\alpha$. Then the angle $AOR = \alpha - \phi$, and the angle $BOR = \alpha + \phi$.

We have then

$$R_1 : R :: \sin(\alpha+\phi) : \sin 2\alpha, \quad \text{or} \quad R_1 = \frac{\sin(\alpha+\phi)}{\sin 2\alpha} R,$$

and

$$R_2 : R :: \sin(\alpha-\phi) : \sin 2\alpha, \quad \text{or} \quad R_2 = \frac{\sin(\alpha-\phi)}{\sin 2\alpha} R.$$

Hence the total friction is

$$F = \left[\sin(\alpha+\phi) + \sin(\alpha-\phi)\right] \frac{R\sin\phi}{\sin 2\alpha}.$$

But $\sin(\alpha+\phi)+\sin(\alpha-\phi) = 2\sin\alpha\cos\phi$, and $\sin 2\alpha = 2\sin\ \ \cos\alpha$ Hence we have

$$F = \frac{R\sin\phi\cos\phi}{\cos\alpha} = \frac{R\sin 2\phi}{2\cos\alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where ϕ is the angle of repose as given by the Table page 192.

The moment of friction with reference to the axis, if r is the radius of the axle, is

$$M = Fr = \frac{Rr\sin 2\phi}{2\cos\alpha}.$$

If the axle is well greased, the angle of repose ϕ is very small and we may take $\sin 2\phi = 2\sin\phi$, also $\mu = \tan\phi = \sin\phi$. In the practical case of a well-greased axle, then, we have

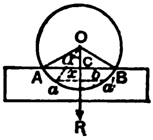
$$F = \mu\frac{R}{\cos\alpha}, \quad M = \mu\frac{Rr}{\cos\alpha},$$

where μ is given by the Table page 192. If the angle α is small,

$\cos\alpha$ may be taken as unity, and F and M are then the same as in the preceding case,

$$F = \mu R, \quad M = \mu Rr.$$

[3. **Axle—New Bearing.**]—When the bearing is new and unworn, the axle touches it at all points.



Let R be the resultant vertical pressure acting at the centre O of the axle. Denote the radius AO of the axle by r, the distance AC by r_1, and let the angle AOC be α. Then the load per unit of horizontal projection is $\dfrac{R}{2r_1}$. Take any element of the surface of the axle at a, distant $ab = x$ from R, and let $Ob = y$. The horizontal projection of this element is dx, and the load sustained by it is $\dfrac{Rdx}{2r_1}$. At a' we have a similar element.

The friction on these two elements is, from the preceding Article,

$$\frac{\sin 2\phi \,.\, Rdx}{2r_1 \cos aOb}.$$

But $\cos aOb = \dfrac{y}{r} = \dfrac{\sqrt{r^2 - x^2}}{r}$, hence the friction for the two elements is

$$\frac{Rr \sin 2\phi}{2r_1} \cdot \frac{dx}{\sqrt{r^2 - x^2}}.$$

Integrating between the limits $x = r_1$ and $x = 0$, we have for the entire friction

$$F = \frac{Rr \sin 2\phi}{2r_1} \sin^{-1} \frac{r_1}{r}.$$

Inserting the value of $r_1 = r \sin\alpha$,

$$F = \frac{R \sin 2\phi}{2} \cdot \frac{\alpha}{\sin\alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where ϕ is the angle of repose as given by the Table page 192.

The moment of the friction with reference to the axis is then

$$M = \frac{Rr \sin 2\phi}{2} \cdot \frac{\alpha}{\sin\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the axle is well greased, the angle of repose ϕ is very small, and we may take $\sin 2\phi = 2 \sin\phi$, also $\mu = \tan\phi = \sin\phi$.

In the practical case of a well-greased axle, then, we have

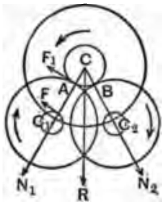
$$F = \mu R \,.\, \frac{\alpha}{\sin\alpha}, \quad M = \mu Rr \frac{\alpha}{\sin\alpha},$$

where μ is given by the Table page 192.

If the angle α is small, we may take $\alpha = \sin\alpha$, and then F and M are the same as in the two preceding cases,

$$F = \mu R, \quad M = \mu Rr.$$

4. **Friction Wheels.**—By the use of friction wheels instead of bearing blocks, the friction of an axle can be greatly diminished.



Thus let the axle AC rest upon the circumferences of the friction wheels AC_1 and BC_2, touching them at the points A and B. The vertical pressure R on the axle C causes the pressures N_1, N_2 at A and B.

Let the angle $ACB = \alpha$. Then

$$N_1 = N_2 = \frac{R}{2\cos\alpha}.$$

If the axles of the friction wheels are well greased, then, as we have seen, the least friction may be written

$$F = \mu(N_1 + N_2) = \frac{\mu R}{\cos\alpha},$$

where μ is given by the Table page 192.

If the radius of the axles of the friction wheels is r, the moment of the friction is

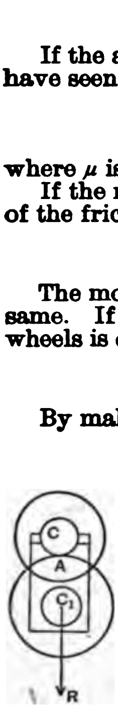
$$Fr = \frac{\mu R r}{\cos\alpha}.$$

The moment of the friction at the points A and B must be the same. If we call this F_1, we have, if the radius of the friction wheels is a,

$$F_1 a = Fr, \quad \text{or} \quad F_1 = \frac{r}{a}F = \frac{r}{a}\cdot\frac{\mu R}{\cos\alpha}.$$

By making α small, we can take $\cos\alpha = 1$, and have

$$F_1 = \frac{r}{a}\cdot\mu R.$$



By taking a large with respect to r, we may thus make the friction F_1 very small. If the axle C rests on bearings, its least friction is μR, as we have seen.

If we have a single friction wheel C_1A, then $\alpha = 0$ and we have accurately

$$F_1 = \frac{r}{a}\mu R.$$

Static Friction of Cords and Chains.—Let a perfectly flexible cord stretched by a weight Q be laid over the edge C of a rigid body ABO, Fig. 1.

FIG. 1.

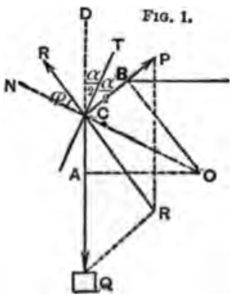
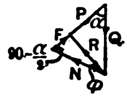


FIG. 2.



Let the force at the other end of the cord be P, and the angle of deviation $DCP = AOB = \alpha$.

Draw CT making the angle $TCP = \frac{\alpha}{2}$, and CN perpendicular to CT. Then when motion is about to begin, the resultant R of P and Q makes the angle of repose ϕ with CN.

If the weight Q is about to sink, the friction F acts opposed to the motion, and we have

$$P + F = Q.$$

We have then, from Fig. 2,

$$F : 2Q \sin\frac{\alpha}{2} :: \sin\phi : \sin\left[90 - \left(\phi - \frac{\alpha}{2}\right)\right],$$

or

$$F = \frac{2Q \sin\frac{\alpha}{2} \sin\phi}{\cos\left(\phi - \frac{\alpha}{2}\right)} = \frac{2Q \sin\frac{\alpha}{2} \sin\phi}{\cos\phi \cos\frac{\alpha}{2} + \sin\phi \sin\frac{\alpha}{2}}.$$

Dividing numerator and denominator by $\cos\phi$, we have, since $\tan\phi = \mu =$ coefficient of static sliding friction, for the friction F_1 when the weight Q is *about to sink*,

$$F_1 = \frac{2\mu Q \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} + \mu \sin\frac{\alpha}{2}} = \frac{2\mu Q \tan\frac{\alpha}{2}}{1 + \mu \tan\frac{\alpha}{2}}. \quad . \quad . \quad . \quad . \quad (1)$$

When the weight Q is *just about to rise*, we have

$$P = Q + F, \quad \text{or} \quad Q = P - F,$$

and hence

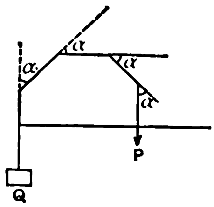
$$F = \frac{2\mu Q \tan\frac{\alpha}{2}}{1 - \mu \tan\frac{\alpha}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the first case, then, when the weight Q is *about to sink*,

$$P_1 = Q - F_1 = \frac{Q\left(1 - \mu \tan\frac{\alpha}{2}\right)}{1 + \mu \tan\frac{\alpha}{2}}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and in the second case, when the weight Q is *about to rise*,

$$P = Q + F = \frac{Q\left(1 + \mu \tan\frac{\alpha}{2}\right)}{1 - \mu \tan\frac{\alpha}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$



If the cord passes over several edges, the force P_1 can be calculated by repeated application of these formulas.

Thus let the number of edges be n and the deviation at each edge be the same and equal to α. When the weight Q is *just about to sink*, the tension of the first portion of the cord is, from (3),

$$P_1 = \frac{Q\left(1 - \mu \tan\frac{\alpha}{2}\right)}{1 + \mu \tan\frac{\alpha}{2}}.$$

That of the second is

$$P_2 = \frac{P_1\left(1-\mu\tan\frac{\alpha}{2}\right)}{1+\mu\tan\frac{\alpha}{2}} = \frac{Q\left(1-\mu\tan\frac{\alpha}{2}\right)^2}{\left(1+\mu\tan\frac{\alpha}{2}\right)^2}.$$

That of the last is

$$P_n = \frac{Q\left(1-\mu\tan\frac{\alpha}{2}\right)^n}{\left(1+\mu\tan\frac{\alpha}{2}\right)^n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If the weight Q, is *just about to rise*, we have simply to interchange P and Q and we have

$$P_n = \frac{Q\left(1+\mu\tan\frac{\alpha}{2}\right)^n}{\left(1-\mu\tan\frac{\alpha}{2}\right)^n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

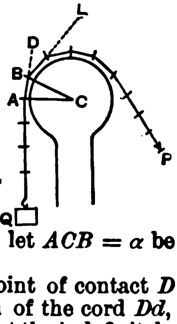
In the first case, when the weight is *about to sink*, we have for the friction

$$F_1 = Q - P_n = Q\left(1 - \frac{\left(1-\mu\tan\frac{\alpha}{2}\right)^n}{\left(1+\mu\tan\frac{\alpha}{2}\right)^n}\right). \quad . \quad . \quad . \quad (7)$$

If the weight is *about to rise*,

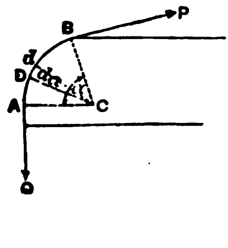
$$F = P_n - Q = Q\left(\frac{\left(1+\mu\tan\frac{\alpha}{2}\right)^n}{\left(1-\mu\tan\frac{\alpha}{2}\right)^n} - 1\right). \quad . \quad . \quad . \quad (8)$$

Formulas (5), (6), (7) and (8) are also applicable to the case of a chain composed of links which is passed round a cylindrical surface, where n is the number of links in contact. If the length of each link is $AB = l$, and the distance CA of the axis A of a link from the centre C is r, we have for the angle of deviation $DBL = ACB = \alpha$,



$$\sin\frac{\alpha}{2} = \frac{l}{2r}, \quad \text{or} \quad \tan\frac{\alpha}{2} = \frac{l}{\sqrt{4r^2 - l^2}}.$$

[If a flexible cord lies in contact with a rough surface, let $ACB = \alpha$ be the arc of contact.



If T is the tension at any point of contact D for the indefinitely small portion of the cord Dd, the friction at this point is dT. Let the indefinitely small angle DCd be $d\alpha$. Then, from equation (1), page 200,

$$dT = \frac{2\mu T\tan\frac{d\alpha}{2}}{1+\mu\tan\frac{\alpha}{2}}.$$

But since $d\alpha$ is indefinitely small, we may take the arc equal to the tangent and disregard $\mu \tan \frac{\alpha}{2}$ with reference to 1. We have then

$$\frac{dT}{T} = \mu d\alpha.$$

Integrating between the limits $\alpha = 0$ and α, we have, since for $\alpha = 0$, $T = Q$, and for $\alpha = \alpha$, $T = P$,

$$\text{logn}\, P = \mu\alpha + \log Q, \quad \text{or} \quad \text{logn}\, \frac{P}{Q} = \mu\alpha.$$

We have then, when motion in the direction of P just begins,

$$P = Q\epsilon^{\mu\alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where $\epsilon = 2.3026$ = base of Naperian system of logarithms.

When motion in the direction of Q just begins, we have, by interchanging P and Q,

$$Q = P\epsilon^{-\mu\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Also, inversely,

$$\alpha = \frac{2.3026(\log P - \log Q)}{\mu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where common logarithms are taken.

If the arc α of the cord is given in degrees instead of radians, we must substitute $\alpha = \frac{\alpha^\circ}{180^\circ}\pi$. If the surface is cylindrical and the number of coils n of the rope is given, we have $\alpha = 2\pi n$.

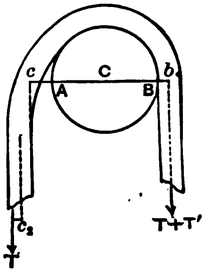
We see from (9) and (10) that the friction of a cord, $F = P - Q$ or $F = Q - P$, upon a surface *does not depend at all upon the radius of curvature*, but only upon the arc of contact α, or upon the number of coils, $2\pi n$, if the surface is cylindrical.

If we take $\mu = \frac{1}{3}$, we have for a cylindrical surface:

for $\frac{1}{4}$ coils, $P = 1.69Q$;
" $\frac{1}{2}$ " $P = 2.85Q$;
" 1 " $P = 8.12Q$;
" 2 " $P = 65.94Q$;
" 4 " $P = 4348.56Q$.

The friction can thus be increased to any amount by increasing the number of coils.]

Rigidity of Ropes.—When a rope is perfectly flexible it offers no resistance to bending. When a rope is not perfectly flexible it offers a resistance by reason of its rigidity when wound *on* to a drum, pulley or axle, though none is offered when it is wound *off*. Thus let a rope whose tension is T be on the point of being wound on to a pulley.



Let $a = \overline{AC} = \overline{BC}$ be the radius of the pulley, and t the thickness of the rope. Then the lever-arm of the axis of the rope on the *off* side is

$$\overline{Cb} = a + \frac{t}{2}.$$

The distance $\overline{Ac}$ from the pulley to the rope

on the *on* side will depend on the kind of rope and will be less as is greater. Thus for *hemp* ropes we can put

$$\overline{Ac} = \frac{c_1}{T},$$

where c_1 is a constant to be determined by experiment for the kind of rope; and for *wire ropes*

$$\overline{Ac} = \frac{c_1\left(a + \frac{t}{2}\right)}{T};$$

that is, $\overline{Ac}$ increases with the lever-arm $a + \frac{t}{2}$ and decreases as T increases.

It is also evident that those fibres farthest out on the *on* side are stretched more than those nearer the pulley. The resultant tension T will therefore act further from the pulley than the central axis of the rope. We denote the distance of T from the central axis by c_2.

Let the tension along the central axis on the *off* side be $T + T'$. Then we have for equilibrium, for *hemp ropes*,

$$T\left(a + \frac{t}{2} + \frac{c_1}{T} + c_2\right) = \left(T + T'\right)\left(a + \frac{t}{2}\right),$$

or

$$T' = \frac{c_1 + c_2 T}{a + \frac{t}{2}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and for *wire ropes*,

$$T\left(a + \frac{t}{2} + \frac{c_1\left(r + \frac{t}{2}\right)}{T} + c_2\right) = \left(T + T'\right)\left(a + \frac{t}{2}\right),$$

or

$$T' = c_1 + \frac{c_2 T}{a + \frac{t}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We have then

$$T \times \overline{Cc} = (T + T')\overline{Cb}, \quad \text{or} \quad \overline{Cc} = \left(1 + \frac{T'}{T}\right)\overline{Cb}. \quad . \quad . \quad (3)$$

The rope can be considered, then, as without rigidity if we increase the lever-arm of the tension on the on-side by the amount $\frac{T'}{T}$.

Hemp Ropes.—For *tarred* hemp ropes experiment gives

$$T' = \frac{100 + 0.222T}{a + \frac{t}{2}} \text{ pounds,}$$

where T is to be taken in pounds and a and t in inches.

For new hemp ropes, *untarred*,

$$T' = \frac{4 + 0.06457T}{a + \frac{t}{2}} \text{ pounds,}$$

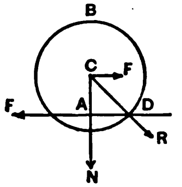
where T is to be taken in pounds and a and t in inches.

Wire Ropes.—For *wire ropes* we have

$$T' = 1.08 + \frac{0.0937T}{a + \frac{t}{2}} \text{ pounds,}$$

where T is to be taken in pounds and a and t in inches.

Static Rolling Friction.—Let ACB be a roller resting on a plane surface. By reason of the pressure N of the roller on the plane, the roller is compressed. Let a force F be applied at the centre C parallel to the plane. When the resultant R of F and N just passes through the edge D of the base, rolling begins and the force F is equal and opposite to the friction.



Let the distance $AD = d$. Then, when rolling is about to begin, the angle ACD is the angle of repose ϕ. Let r be the radius. Since the compression is small compared to the radius, we have $\tan\phi = \frac{d}{r} = \mu =$ coefficient of static rolling friction. Hence for equilibrium $Fr = Nd$, or

$$F = \mu N = \frac{d}{r}N.$$

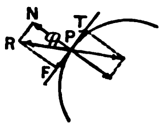
The distance d depends on the materials in contact.

The theory of rolling friction is not yet well established and but few experiments upon it have been made.

In all practical cases of rolling, we usually have to do with axle friction, which has already been discussed (page 196).

[Equilibrium of a Body at Any Point of a Rough Curve or Surface—General Equations.]—If a body acted upon by any number of forces F_1, F_2, etc., applied at different points, is at rest at any point of a rough curve or surface, we may treat it as a particle placed at that point (page 188).

The reaction R at that point must be equal and opposite to the resultant of all the other forces acting upon the body.



The curve or surface can then be replaced by its reaction R at the point P. For limiting equilibrium the reaction R must make an angle with the normal to the curve or surface at the point P equal to the angle of repose ϕ, given by

$$\tan\phi = \mu, \qquad (1)$$

where μ is the coefficient of static sliding friction.

If R makes an angle with the normal less than ϕ, we have non-limiting equilibrium (page 189). If equal to ϕ, we have limiting equilibrium, and sliding is about to begin.

Let the algebraic sum of the components along the co-ordinate axes of all the forces F_1, F_2, etc., *not including the friction and the reaction R at the point P*, be F_x, F_y, F_z. Then if the direction-angles are $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, etc., we have

$$F_x = F_1 \cos\alpha_1 + F_2 \cos\alpha_2 + \ldots = \Sigma F \cos\alpha;$$

$$F_y = F_1 \cos\beta_1 + F_2 \cos\beta_2 + \ldots = \Sigma F \cos\beta;$$

$$F_z = F_1 \cos\gamma_1 + F_2 \cos\gamma_2 + \ldots = \Sigma F \cos\gamma.$$

1. Equilibrium of a Body at Any Point of a Rough Curve.—Let the co-ordinates of the point P be x, y, z, and ds be an element of the curve.

Then the direction-cosines of the tangent to the curve at the point P are $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, and we have for the force T tangential to the curve

$$T = F_x\frac{dx}{ds} + F_y\frac{dy}{ds} + F_z\frac{dz}{ds}.$$

The reaction R makes with the normal an angle whose sine is $\frac{T}{R}$. For equilibrium this angle must be less than the angle of repose ϕ, or $\frac{T}{R}$ is less than $\sin\phi$. Hence the condition for non-limiting equilibrium is

$$\frac{F_xdx + F_ydy + F_zdz}{Rds} < \sin\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or, since $\sin^2\phi = \frac{\mu^2}{1+\mu^2}$,

$$\left(\frac{F_xdx + F_ydy + F_zdz}{Rds}\right)^2 < \frac{\mu^2}{1+\mu^2}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If then

$$\frac{F_xdx + F_ydy + F_zdz}{Rds} = \pm\sin\phi = \pm\sqrt{\frac{\mu^2}{1+\mu^2}}, \quad . \quad . \quad . \quad (4)$$

we have limiting equilibrium, and the body is upon the point of sliding.

The force T for equilibrium is always equal and opposite to the friction F, or $T + F = 0$. Hence

$$F_x\frac{dx}{ds} + F_y\frac{dy}{ds} + F_z\frac{dz}{ds} + F = 0.$$

If we multiply by ds, we have

$$F_xdx + F_ydy + F_zdz + Fds = 0,$$

which is the *principle of virtual work* (page 159).

2. Equilibrium of a Body at Any Point of a Rough Surface.—Let the equation of the surface be $u = 0$, where u is a function of x, y, z.

For convenience of notation let

$$\frac{du}{dx} = U, \quad \frac{du}{dy} = V, \quad \frac{du}{dz} = W, \quad \text{and} \quad U^2 + V^2 + W^2 = Q^2.$$

Then the direction-cosines of the normal to the surface at the point (x, y, z) are

$$\frac{U}{Q}, \quad \frac{V}{Q}, \quad \frac{W}{Q}.$$

The resolved part of R along the normal is then

$$N = F_x\frac{U}{Q} + F_y\frac{V}{Q} + F_z\frac{W}{Q}.$$

The reaction R makes with the normal an angle whose cosine is $\frac{N}{R}$.

For equilibrium this angle must be less than the angle of repose ϕ, or $\frac{N}{R}$

is greater than $\cos\phi$. Hence the condition for non-limiting equilibrium is

$$\frac{F_x\frac{du}{dx} + F_y\frac{du}{dy} + F_z\frac{du}{dz}}{R\sqrt{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2}} > \cos\phi, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

or, since $\cos^2\phi = \frac{1}{1+\mu^2}$,

$$\frac{\left(F_x\frac{du}{dx} + F_y\frac{du}{dy} + F_z\frac{du}{dz}\right)^2}{R^2\left[\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2\right]} > \frac{1}{1+\mu^2}. \quad . \quad . \quad . \quad (6)$$

If then

$$\frac{F_x\frac{du}{dx} + F_y\frac{du}{dy} + F_z\frac{du}{dz}}{R\sqrt{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2}} = \pm\cos\phi = \pm\sqrt{\frac{1}{1+\mu^2}}, \quad . \quad (7)$$

we have limiting equilibrium, and the body is upon the point of sliding.

Let the point P be moved in any direction along the surface through the indefinitely small distance ds, and dx, dy, dz be the projections of this distance on the axes. Then the direction-cosines of the tangent at the point P are $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$. The tangential force T is equal and opposite to the friction F, or $T = -F$. We have then

$$F_x = N\frac{\frac{du}{dx}}{Q} - F\frac{dx}{ds}, \quad F_y = N\frac{\frac{du}{dy}}{Q} - F\frac{dy}{ds}, \quad F_z = N\frac{\frac{du}{dz}}{Q} - F\frac{dz}{ds}.$$

If we multiply the first of these by dx, the second by dy, the third by dz, add the results and reduce by the equations $dx^2 + dy^2 + dz^2 = ds^2$ and

$$\left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy + \left(\frac{du}{dz}\right)dz = 0,$$

which is the total differential of the equation $u = 0$ of the surface, we obtain

$$F_x dx + F_y dy + F_z dz + F ds = 0,$$

which is the *principle of virtual work* (page 159).

Stable, Unstable, Neutral and Indifferent Equilibrium.—A body in equilibrium is said to be in **stable equilibrium** when for *every* possible indefinitely small displacement which it can receive it tends to return to its original position.

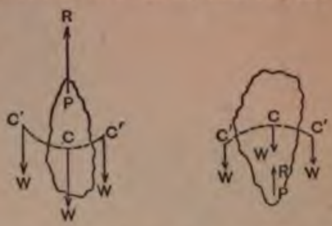
When for *any one* possible indefinitely small displacement it tends to move still farther away from its original position of equilibrium, it is in **unstable equilibrium**.

Cases occur in which the equilibrium of a body is stable for some displacements and unstable for others. It is then, by definition, in unstable equilibrium.

If the body remains in equilibrium for all possible indefinitely small displacements, it is in **neutral equilibrium**. Neutral equilibrium may be stable or unstable.

If the body remains in equilibrium for all possible displacements, *large or small*, it is in **indifferent equilibrium**.

Thus let a heavy body be supported at a fixed point P, so that it can only rotate about P. Let the reaction at P be R, and let the weight W act at the centre of mass C. Then for equilibrium, if R and W are the only forces acting upon the body, the reaction R must be equal and opposite to W and act in the same line.



We have then two possible positions of equilibrium: one when C is below P, and one when C is above P.

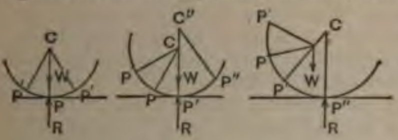
Now for every possible indefinitely small displacement of rotation about P, the point C moves in the surface of a sphere $C'CC'$ of radius PC, and W remains unchanged in magnitude and direction.

Therefore in the first case, when C is below P, we have for every possible displacement a couple R and W which always tends to make the body return to its original position of equilibrium, and the body is in stable equilibrium.

In the second case, when C is above P, we have for every possible displacement a couple R and W which always tends to make the body move still farther from its original position of equilibrium, and the body is in unstable equilibrium.

If the points P and C coincide, then for every possible displacement, large or small, the body remains in equilibrium, and the body is therefore in indifferent equilibrium.

Again, let a heavy body bounded by a convex surface rest in equilibrium on a plane surface, and let the centre of mass C *coincide with the centre of curvature*. Then the reaction R acts at the point of contact P, is equal and opposite to the weight W and acts in the same straight line.



If the body can have *rolling motion only*, any indefinitely small arc PP' is circular. Hence for any possible indefinitely small displacement produced by rolling, the body remains still in equilibrium. Its original position is therefore one of neutral equilibrium.

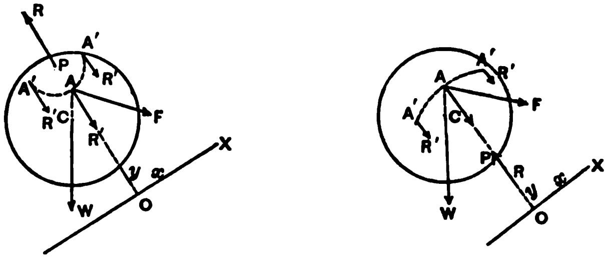
If now the body be rolled still farther through an indefinitely small arc, so that P'' comes in contact with the plane, then, if the radius of curvature CP is less than $C''P''$, the equilibrium is evidently stable; if greater, unstable. The original position of neutral equilibrium is therefore stable neutral when the radius of curvature CP is a minimum, and unstable neutral when it is a maximum. When it is not a minimum or maximum, the neutral equilibrium is stable for displacement in one direction and unstable for displacement in the other direction—that is, unstable neutral, according to definition (page 206).

Criterion for Stable, Unstable, Neutral and Indifferent Equilibrium.—Every displacement of a body consists in general of two displacements, one of translation and one of rotation. Now for an indefinitely small displacement of translation, a body which under the action of certain forces is in equilibrium before the displacement is also in equilibrium after, if the forces act at the same points, because their magnitudes and directions are unchanged by the displacement.

Thus if a body can only slide on a plane surface and touches it in more than two points not in a straight line, it can only receive motion of translation and its equilibrium is indifferent.

We have then only to determine the conditions for stable, unstable, neutral and indifferent equilibrium in the case of rotation.

Let P be the point about which the body can rotate, and R the reaction at that point. Let the weight W act at the centre of mass C, and let the resultant of all the other forces acting upon the body



be F. Then for equilibrium the lines of direction of W, F and R must intersect in a point A which lies in the vertical through the centre of mass C, and the resultant R' of W and F must be equal and opposite to R and act in the same straight line.

For every possible indefinitely small displacement of rotation about P the point A moves in the surface of a sphere $A'AA'$ of radius PA, and R' remains unchanged in magnitude and direction. If the body has a displacement of translation as well as of rotation about P, the locus $A'AA'$ of the point A is no longer a spherical, but is still a curved surface.

We see at once from the figures that for any displacement for which the projection of AA' along R' is opposite in direction to R', or for which the *work of R' is negative* (page 158), the body tends to return to its original position of equilibrium. For any displacement for which the *work of R' is positive*, the body tends to move away from its original position of equilibrium.

Let us take the axis of Y parallel to R', and the direction of R' as *downwards*.

Then if we draw a line OX at right angles to R' at any distance $AO = y$ *below* A, we see from the figures that when $AO = y$ is a minimum, the equilibrium is stable. When $AO = y$ is a maximum, the equilibrium is unstable for all possible displacements. When $AO = y$ is neither a maximum nor a minimum, the equilibrium is stable for some displacemements and unstable for others; that is, unstable according to definition (page 206).

In general, then:

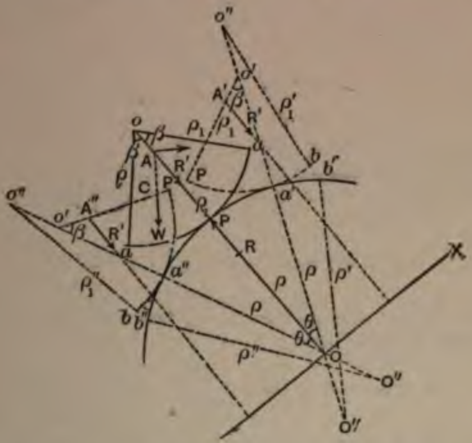
A body is in stable equilibrium when for all possible indefinitely small displacements the work of the resultant of all the forces except the reaction is negative. If for any or all possible indefinitely small displacements this work is positive, the equilibrium is unstable. If it is zero for all possible indefinitely small displacements, the equilibrium is neutral. If it is zero for all possible displacements, large or small, the equilibrium is indifferent.

Or if we take the axis of Y parallel to R' and the direction of R' as downwards, and draw OX at right angles to R' at any distance $AO = y$ *below* A, we have the criterion:

A body is in stable equilibrium when for all possible indefinitely small displacements $AO = y$ is a minimum. If $AO = y$ is a maxi-

mum, the equilibrium is unstable for all possible indefinitely small displacements. If $AO = y$ is neither a maximum nor a minimum, the equilibrium is stable for some displacements and unstable for others, that is, unstable. If for all possible indefinitely small displacements $AO = y$ remains constant, the equilibrium is neutral. If for all possible displacements, large or small, $AO = y$ remains constant, the equilibrium is indifferent.

Stability in Rolling Contact.—As an application of the preceding, let us investigate the equilibrium of a heavy body aPa bounded by a convex surface resting upon a rough body $a''Pa'$ also with a convex surface, and subject to displacement due to *rolling only.*



Let O be the centre of curvature of the fixed body, and o the centre of curvature of the rolling body, so that the radius of curvature of the fixed body at the point of contact P is $OP = \rho$, and the radius of curvature of the rolling body at the point of contact P is $oP = \rho_1$.

Through O draw a plane OX at right angles to OP.

Let the reaction at P be R, the weight at the rolling body be W acting at its centre of mass C, and the resultant of all other forces acting upon the rolling body be F.

Then for equilibrium the lines of direction of W, F and R must intersect in a point A, which lies in a vertical through the centre of mass C, and the resultant R' of W and F must be equal and opposite to the reaction R at the point of contact P and act in the same straight line.

The point A may or may not be in the radius oP. If it is not in the radius oP, then, provided R' passes through P and makes an angle with oP less than the angle of repose, there will be equilibrium (page 188). But in such case it is evident that if for rolling in one direction in the plane of R' and oP the distance AO of A from OX increases, for rolling in the opposite direction this distance will decrease. The rolling body is then, according to definition (page 206), *always in unstable equilibrium* if A is not in the radius oP, and there is no need of discussion.

If, however, A is in the radius oP, the equilibrium will be stable, unstable or neutral, according as the distance AO of A from

OX increases for *all* possible indefinitely small displacements, decreases for *any or all* possible indefinitely small displacements, or remains constant for *all* possible indefinitely small displacements (page 208).

Let the points a, P, a of the rolling body be consecutive. Then the arc aPa is circular, its radius is ρ_1, the arcs $aP = aP$, and the angles $aoP = \beta$ are indefinitely small.

Let the body roll in either direction, so that the points a, a become points of contact at a' and a''. Then the arc $a''Pa'$ is circular, its radius is ρ, the arcs $a''P = a'P$, and the angles $a''OP = a'OP = \theta$ are indefinitely small.

Let the distance $Ao = c$. Then $PA = \rho_1 - c$. When the body rolls into its new position, the point A passes to A' or A'', and we have $A'o' = A''o'' = Ao = c$, or $PA' = PA'' = PA = \rho_1 - c$. Also, since the arcs Pa and Pa' or Pa'' are equal,

$$\rho\theta = \rho_1\beta, \quad \text{or} \quad \beta = \frac{\rho}{\rho_1}\theta, \quad \text{or} \quad \theta + \beta = \left(1 + \frac{\rho}{\rho_1}\right)\theta. \quad . \quad . \quad (1)$$

Now the distance AO of A from OX is $\rho + \rho_1 - c$. The distance of A' or A'', or the distance of A after indefinitely small displacement, from OX, is $(\rho + \rho_1) \cos \theta - c \cos (\theta + \beta)$, or, inserting the value of $(\theta + \beta)$ from (1),

$$(\rho + \rho_1) \cos \theta - c \cos \left(1 + \frac{\rho}{\rho_1}\right)\theta. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If we replace the cosines in (2) by the first two terms of their equivalent series, (2) becomes

$$\rho + \rho_1 - c - (\rho + \rho_1)\left[1 - c\frac{\rho + \rho_1}{\rho_1^2}\right]\frac{\theta^2}{2}. \quad . \quad . \quad . \quad . \quad (3)$$

Hence the equilibrium is stable, unstable or neutral according as (3) is greater, less or equal to $\rho + \rho_1 - c$.

When (3) is greater than $\rho + \rho_1 - c$, the coefficient of θ^2 must be positive, or

$$c\frac{\rho + \rho_1}{\rho_1^2} > 1.$$

The condition for *stable equilibrium* is then, since $PA = \rho_1 - c$,

$$c > \frac{\rho_1^2}{\rho + \rho_1}, \quad \text{or} \quad \frac{1}{PA} > \frac{1}{\rho} + \frac{1}{\rho_1}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The condition for *unstable equilibrium* is

$$c < \frac{\rho_1^2}{\rho + \rho_1}, \quad \text{or} \quad \frac{1}{PA} < \frac{1}{\rho} + \frac{1}{\rho_1}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The condition for *neutral equilibrium* is

$$c = \frac{\rho_1^2}{\rho + \rho_1}, \quad \text{or} \quad \frac{1}{PA} = \frac{1}{\rho} + \frac{1}{\rho_1}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In order to find whether the neutral equilibrium is stable or unstable, let O'' and o'' be the centres of curvature for the indefinitely small arcs $a'b'$ or $a''b''$ and $a'b$ or $a''b$, and let the radii of curvature be ρ' or ρ'' and ρ_1' or ρ_1''.

Then, proceeding just as before, we find for the conditions of *stable* neutral equilibrium

$$\frac{1}{\rho}+\frac{1}{\rho_1}>\frac{1}{\rho'}+\frac{1}{\rho_1'} \quad \textit{and also} \quad >\frac{1}{\rho''}+\frac{1}{\rho_1''}. \quad . \quad . \quad . \quad (7)$$

For *unstable* neutral equilibrium we have

$$\frac{1}{\rho}+\frac{1}{\rho_1}<\frac{1}{\rho'}+\frac{1}{\rho_1'} \quad \textit{or} \quad <\frac{1}{\rho''}+\frac{1}{\rho_1''}. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If the first of (7) and second of (8) are fulfilled, we have stable neutral equilibrium for displacement towards the right in the figure, and unstable neutral equilibrium for displacement towards the left, and *vice versa* if the second of (7) and first of (8) are fulfilled. In either case the neutral equilibrium is unstable according to definition (page 206).

We can also find conditions (4), (5) and (6) as follows:

The line of direction of R' after displacement must fall between P and a' or P and a'' for stable equilibrium, outside of Pa or Pa'' for unstable, and pass through a' or a'' for neutral equilibrium.

Hence we have for stable equilibrium

$$\rho \sin \theta > (\rho+\rho_1)\sin\theta - c\sin(\theta+\beta).$$

Since θ and β are indefinitely small, we can put the arcs in place of their sines. Putting then $\theta+\beta=\left(1+\frac{\rho}{\rho_1}\right)\theta$, as given by (1), we have

$$\rho\theta > (\rho+\rho_1)\theta - c\left(1+\frac{\rho}{\rho_1}\right)\theta, \quad \text{or} \quad c>\frac{\rho_1^2}{\rho+\rho_1}.$$

Hence we obtain, as before, conditions (4), (5) and (6).

Special Cases.—Conditions (4), (5), (6), (7) and (8) are general. Thus if the concavity of either surface be turned the other way, we shall obtain the same results, except that the sign of the corresponding radius of curvature will be changed.

Surfaces Spherical.—If one of the surfaces is spherical, we have $\rho=\rho'=\rho''$, or $\rho_1=\rho_1'=\rho_1''$. If both surfaces are spherical, we have $\rho=\rho'=\rho''$ and $\rho_1=\rho_1'=\rho_1''$. If in the latter case the equilibrium is neutral, we have from (7)

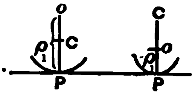
$$\frac{1}{\rho}+\frac{1}{\rho_1}=\frac{1}{\rho'}+\frac{1}{\rho_1'}=\frac{1}{\rho''}+\frac{1}{\rho_1''};$$

that is, the neutral equilibrium is indifferent as long as R' does not change in direction and the angle θ is less than the angle of repose ϕ, or β is less than $\frac{\rho}{\rho_1}\phi$. For $\theta>\phi$ or $\beta>\frac{\rho}{\rho_1}\phi$ there is sliding.

Either Surface Plane.—If either surface is plane, its radius of curvature becomes indefinitely great and the corresponding $\frac{1}{\rho}$ or $\frac{1}{\rho_1}$ is zero.

Weight Only Considered.—If the only forces acting upon the rolling body are its weight W and the reaction R, we have $F=0$,

$R' = W$ acting vertically. The centre of mass C then coincides with A, and we have PC in place of PA in (4), (5) and (6).



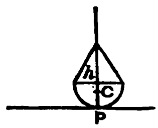
Heavy Body on Plane Surface.—In this case we have $\frac{1}{\rho} = 0$, and $PC < \rho_1$ for stable, $PC > \rho_1$ for unstable, equilibrium.

If $PC = \rho_1$, or the centre of mass coincides with the centre of curvature, the equilibrium is neutral. In such case we have, from (7), stable equilibrium when ρ_1 is less than ρ_1' and ρ_1'', that is, when ρ_1 or the radius of curvature is a *minimum*. When ρ_1 is not a minimum nor a maximum, the neutral equilibrium is stable for some displacements and unstable for others, or unstable according to definition (page 206). If ρ_1 is a maximum, the neutral equilibrium is unstable for all possible indefinitely small displacements. If the radius of curvature is constant, the neutral equilibrium holds for all possible displacements large or small, we have a homogeneous sphere rolling on a plane, and the equilibrium is indifferent.

EXAMPLES.

(1) *A body made up of a cone and a hemisphere having a common base rests with the axis vertical on a rough horizontal plane. Find the greatest height of the cone for stable equilibrium.*

Ans. Let h be the height of the cone, r the radius of the hemisphere, and C the centre of mass. The height required is that height for which $PC = r$.



The volume of the hemisphere is $\frac{2}{3}\pi r^3$. The volume of the cone is $\frac{1}{3}\pi r^2 h$. The centre of mass of the hemisphere is at a distance above P equal to $\frac{5}{8}r$ (page 422). The centre of mass of the cone is at a distance above P equal to $r + \frac{h}{4}$ (page 420). We have then

$$PC = \frac{\frac{2}{3}\pi r^3 \times \frac{5}{8}r + \frac{\pi r^2 h}{3} \times \left(r + \frac{h}{4}\right)}{\frac{2}{3}\pi r^3 + \frac{\pi r^2 h}{3}} = r, \quad \text{or} \quad h = r\sqrt{3}.$$

(2) *A prolate spheroid rests with its axis horizontal on a rough horizontal plane. Show that for a rolling displacements in its equatorial plane the equilibrium is indifferent, and for rolling displacements in the vertical plane through the axis it is stable.*

(3) *A right circular cylinder of radius r rests with its axis horizontal on a fixed rough sphere of radius R greater than r. Show that for rolling displacements the equilibrium is stable or unstable, according as the plane of displacement makes an angle with the vertical plane through the axis of the cylinder whose sine is less or greater than* $\sqrt{1 - \frac{r}{R}}$.

Ans. Let ρ be the radius of curvature of the rolling curve at the point of contact. Then the condition for stable equilibrium is

$$\frac{1}{r} > \frac{1}{R} + \frac{1}{\rho}.$$

Let the plane of displacement make the angle θ with the vertical plane through the axis of the cylinder The rolling curve is then an ellipse whose semi-minor axis is r and whose semi-major axis is $\dfrac{r}{\sin\theta}$. The radius of curvature at the point of contact, that is, at the vertex of the minor axis, is

$$\rho = \frac{\left(\dfrac{r}{\sin\theta}\right)^2}{r} = \frac{r}{\sin^2\theta}.$$

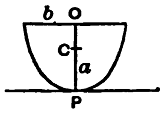
Hence for stable equilibrium

$$\frac{1}{r} > \frac{1}{R} + \frac{\sin^2\theta}{r}, \quad \text{or} \quad \sin\theta < \sqrt{1 - \frac{r}{R}}.$$

(4) *A prolate hemispheroid rests with its vertex on a rough horizontal plane. Show that for rolling displacement the equilibrium is stable or unstable according as the eccentricity of the generating ellipse is less or greater than* $\sqrt{\dfrac{3}{8}}$.

Ans. Let a be the semi-major and b the semi-minor axis. Then the distance OC to the centre of mass (page 41) is

$$OC = \frac{3}{4}\frac{(2a-a)^2}{3a-a} = \frac{3}{8}a.$$



The distance PC then is $\frac{5}{8}a$. The radius of curvature at P is $\rho = \dfrac{b^2}{a}$. We have then for stable equilibrium

$$\frac{1}{PC} > \frac{1}{\rho}, \quad \text{or} \quad \frac{8}{5a} > \frac{a}{b^2}, \quad \text{or} \quad \frac{b^2}{a^2} > \frac{5}{8}.$$

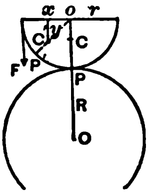
But the eccentricity of the generating ellipse is

$$e = \sqrt{1 - \frac{b^2}{a^2}}.$$

Hence for stable equilibrium $e < \sqrt{\dfrac{3}{8}}$.

(5) *A solid homogeneous hemisphere of radius r and weight W rests in neutral equilibrium on the top of a fixed sphere of radius R. Show that $R = \frac{5}{3}r$. If now a weight F is fastened to any point in the rim of the hemisphere, show that if $F = \frac{18}{55}W$, the hemisphere can still rest in neutral equilibrium at the highest point of the sphere, and that the neutral equilibrium is indifferent for all angular displacements of the hemisphere less than $\frac{5}{8}\phi$, where ϕ is the angle of repose. Also that the radius through the point of contact in the second case makes an angle with the radius in the first case whose tangent is $\frac{48}{55}$.*

Ans. In the first case we have $PC = \frac{5}{8}r$, and the radius of curvature is r,



and PC lies in the axis of symmetry. We have then for neutral equilibrium

$$\frac{1}{PC} = \frac{1}{R} + \frac{1}{r}, \quad \text{or} \quad \frac{8}{5r} = \frac{1}{R} + \frac{1}{r};$$

hence $R = \frac{5}{3}r$. If now we attach the weight F to the rim, the new centre of mass, C', will be at a horizontal distance x from o given by

$$x = \frac{Fr}{W+F},$$

and at a vertical distance y below o given by

$$y = \frac{\frac{3}{8}Wr}{W+F}.$$

The distance $P'C'$ is then given by $P'C' = r - \sqrt{x^2+y^2}$, or

$$P'C' = r - \sqrt{\frac{F^2r^2 + \frac{9}{64}W^2r^2}{(W+F)^2}}.$$

If then we place the hemisphere so that P' is in contact at P, there will be neutral equilibrium when

$$\frac{1}{P'C'} = \frac{1}{R} + \frac{1}{r} = \frac{8}{5r}.$$

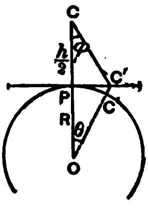
Inserting the value of $P'C'$ and reducing, we obtain $F = \frac{18}{55}W$.

The tangent of the angle POP' is $\frac{x}{y} = \frac{8F}{3W} = \frac{48}{55}$.

Since both surfaces are spherical and equilibrium neutral, we have (page 210) indifferent equilibrium as long as $\beta < \frac{R}{r}\phi$, or $\beta < \frac{5}{3}\phi$.

(6) *A cylinder rests in equilibrium with the centre of its base on the highest point of a fixed and rough sphere. The altitude and diameter of the base of the cylinder are each equal in length to a quadrant of a great circle of the sphere. Find the greatest angle through which the cylinder may be made to rock without falling off.*

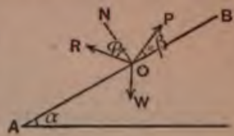
Ans. Let C be the centre of mass of the cylinder, and O the centre of the fixed sphere. Then $PC = \frac{h}{2}$ and $OP = R$. When the cylinder rocks let the points C' come in contact. Then $PC' = R\theta$. If the cylinder is on the point of sliding, the angle PCC' must be equal to the angle of repose ϕ. Hence $\tan\phi = \frac{R\theta}{\frac{h}{2}}$, or $\theta = \frac{h\tan\phi}{2R}$. But we have $\frac{\pi R}{2} = h$.



Therefore $\theta = \frac{\pi}{4}\tan\phi$.

Since $\frac{1}{PC} = \frac{2}{h} > \frac{1}{R} = \frac{\pi}{2h}$, the equilibrium is stable.

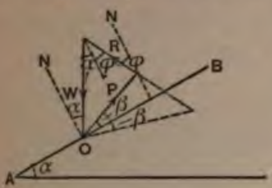
(7) *A body of weight W is placed upon a rough inclined plane which makes an angle α with the horizontal, and is acted upon by a force P which makes the angle β with the plane. Find the conditions of equilibrium.* (For smooth plane see Ex. 1, page 172.)



Ans. Consider the body as a particle placed at any point O on the plane (page 169). We have acting upon the particle the weight W, the force P and the reaction of the plane R, which makes the angle of repose ϕ with the normal to the plane.

Let the angle $BOP = \beta$ be positive when above the plane, and negative when below the plane.

1. Body on the Point of Motion up the Plane.—In this case the component of P along the plane must act up the plane, and the component of R along the plane or the friction must act down the plane, since friction always acts opposite to the direction in which motion tends to take place.



Since W, P and R are in equilibrium, their line representatives laid off in order the same way round make a triangle (page 62).

We have then directly from the figure

$$R : W :: \sin[90 - (\beta + \alpha)] : \sin[90 + (\beta - \phi)].$$

Hence

$$R = \frac{\cos(\beta + \alpha)}{\cos(\beta - \phi)} W.$$

Let the normal pressure of the plane be N and the friction be F.
Then we have

$$F = R \sin\phi = \frac{\cos(\beta + \alpha)}{\cos(\beta - \phi)} W \sin\phi, \qquad N = R\cos\phi = \frac{\cos(\beta + \alpha)}{\cos(\beta - \phi)} W \cos\phi.$$

We also have directly from the figure

$$P : W :: \sin(\alpha + \phi) : \sin[90 + (\beta - \phi)].$$

Hence

$$P = \frac{\sin(\alpha + \phi)}{\cos(\beta - \phi)} W = \frac{\sin\alpha + \mu\cos\alpha}{\cos\beta + \mu\sin\beta} W,$$

where $\mu = \tan\phi$ is the coefficient of static sliding friction.

We see at once from the figure and from the preceding equations that when $\beta = +(90 - \alpha)$ we have R, N and F zero and P and W equal and opposite. For any greater value of positive β, R is negative and there is no equilibrium possible. For negative β we must evidently have β less than $90 - \phi$. If β is greater than this, R is negative and there is no equilibrium. *The preceding equations hold good, then, for all values of β between $-(90 - \varphi)$ and $+(90 - \alpha)$.*

The force P is a *minimum* when $\cos(\beta - \phi)$ is a maximum or when $\beta = +\phi$. This minimum value of P is then

$$P = W\sin(\alpha + \phi).$$

Again, we can resolve P into $P\cos\beta$ along the plane and $P\sin\beta$ normal to the plane. We can also resolve W into $W\sin\alpha$ along the plane and $W\cos\alpha$ normal to the plane. Let N be the normal pressure of the plane. Then for equilibrium

$$N + P\sin\beta - W\cos\alpha = 0, \quad \text{or} \quad N = W\cos\alpha - P\sin\beta.$$

The friction is then

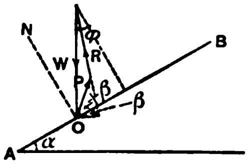
$$F = \mu N = \mu W \cos\alpha - \mu P \sin\beta.$$

This friction always acts opposite to the direction in which motion tends to take place. We have then in the present case, for equilibrium,

$$P\cos\beta - F - W\sin\alpha = 0.$$

Inserting the value for F and reducing, we obtain the same value for P as before. The student should also solve by virtual work.

2. Body on the Point of Motion Down the Plane—α greater than ϕ.—If α *is greater than* ϕ, the body will slide down the plane unless prevented.



In this case the component of P along the plane must act up the plane, and the component of R along the plane, or the friction, must also act up the plane, since friction always acts opposite to the direction in which motion tends to take place.

We have then directly from the figure

$$R : W :: \sin[90 - (\beta + \alpha)] : \sin[90 + (\beta + \phi)].$$

Hence

$$R = \frac{\cos(\beta + \alpha)}{\cos(\beta + \phi)} W;$$

$$F = R\sin\phi = \frac{\cos(\beta + \alpha)}{\cos(\beta + \phi)} W\sin\phi, \quad N = R\cos\phi = \frac{\cos(\beta + \alpha)}{\cos(\beta + \phi)} \cos\phi.$$

Also

$$P : W :: \sin(\alpha - \phi) : \sin[90 + (\beta + \phi)],$$

or

$$P = \frac{\sin(\alpha - \phi)}{\cos(\beta + \phi)} W = \frac{\sin\alpha - \mu\cos\alpha}{\cos\beta - \mu\sin\beta} W.$$

We see again from the figure that when $+\beta$ is greater than $90 - \alpha$, R is negative. Also when $-\beta$ is greater than $90 + \phi$, R is negative. *The values of R, F, N and P hold good, then, for values of β between* $-(90 + \phi)$ *and* $+(90 - \alpha)$.

The force P is a *minimum* when $\cos(\beta + \phi)$ is a maximum or when $\beta = -\phi$. This minimum value of P is then

$$P = W\sin(\alpha - \phi)$$

As long as we have, *for α greater than ϕ,*

$$P > \frac{\sin(\alpha - \phi)}{\cos(\beta + \phi)} W,$$

where $\beta < -(90 +)\phi$ and $< +(90 - \alpha)$, and at the same time have

$$P < \frac{\sin(\alpha + \phi)}{\cos(\beta - \phi)} W,$$

where $\beta < -(90 - \phi)$ and $< +(90 - \alpha)$, the body will neither be on the point of moving down or up, and we have non-limiting equilibrium.

Again, we have as before for the friction

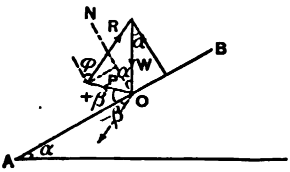
$$F = \mu W\cos\alpha - \mu P\sin\beta,$$

and for motion down the plane

$$P\cos\beta + F - W\sin\alpha = 0.$$

Substituting the value of F and reducing, we obtain the same value for P as before. The student should solve also by virtual work.

3. Body on the Point of Motion Down the Plane—α less than ϕ.—If α is *less than* ϕ, the body will not slide down unless acted upon by some force P.



In this case the component of P along the plane must act down the plane, and the component of R along the plane, or the friction, must act up the plane, since friction always acts opposite to the direction in which motion tends to take place.

We have then directly from the figure, if we take the angle $\beta = AOP$ positive above the plane and negative below,

$$R : W :: \sin[90 - (\beta - \alpha)] : \sin[90 + (\beta - \phi)].$$

Hence

$$R = \frac{\cos(\beta - \alpha)}{\cos(\beta - \phi)} W;$$

$$F = R \sin\phi = \frac{\cos(\beta - \alpha)}{\cos(\beta - \phi)} W \sin\phi, \quad N = R\cos\phi = \frac{\cos(\beta - \alpha)}{\cos(\beta - \phi)} W \cos\phi.$$

Also

$$P : W :: \sin(\phi - \alpha) : \sin[90 + (\beta - \phi)],$$

or

$$P = \frac{\sin(\phi - \alpha)}{\cos(\beta - \phi)} W = \frac{\mu\cos\alpha - \sin\alpha}{\cos\beta + \mu\sin\beta} W.$$

We see from the figure that if we take β from OA positive above and negative below the plane, $+\beta$ cannot be greater than 90. Also when $-\beta$ is greater than $90 - \phi$, R is negative. *The values of R, F, N and P hold good, then, for values of β between $-(90 - \phi)$ and $+90°$.*

The force P is a minimum when $\cos(\beta - \phi)$ is a maximum or when $\beta = +\phi$. This minimum value of P is then

$$P = W\sin(\phi - \alpha).$$

As long as we have, *for α less than ϕ*,

$$P < \frac{\sin(\phi - \alpha)}{\cos(\beta - \phi)} W,$$

where $\beta < +(90 + \alpha)$ and $< -(90 - \phi)$, and at the same time have

$$P < \frac{\sin(\alpha + \phi)}{\cos(\beta - \phi)} W,$$

where $\beta < +(90 - \alpha)$ and $< -(90 - \phi)$, the body will neither be on the point of moving up or down and we have non-limiting equilibrium.

Again, we have, as before, for the friction

$$F = \mu W\cos\alpha - \mu P\sin\beta,$$

and for motion down the plane

$$-P\cos\beta + F - W\sin\alpha = 0.$$

Substituting the value of F and reducing, we obtain the same value for P as before. The student should solve also by virtual work.

(8) *A body of weight W is placed in contact with the under side of a rough inclined plane which makes an angle α with the horizontal, and is acted upon by a force P which makes an angle β with the plane. Find the conditions of equilibrium.* (For smooth plane see Ex. (2), page 174.)

Ans. 1st. Body on the point of motion up the plane:

$$R = -\frac{\cos(\beta+\alpha)}{\cos(\beta+\phi)}W, \quad N = -\frac{\cos(\beta+\alpha)}{\cos(\beta+\phi)}W\cos\phi;$$

$$F = -\frac{\cos(\beta+\alpha)}{\cos(\beta+\phi)}W\sin\phi;$$

$$P = -\frac{\sin(\phi-\alpha)}{\cos(\beta+\phi)}W = \frac{\sin\alpha-\mu\cos\alpha}{\cos\beta-\mu\sin\beta}W.$$

When $\alpha > \phi$, $\beta > +(90-\alpha)$ and $< +(90-\phi)$.
When $< \phi$, $\beta > +(90-\phi)$ and $< +(90-\alpha)$.

2d. Body on the point of motion down the plane:

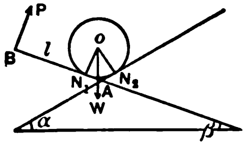
$$R = -\frac{\cos(\beta-\alpha)}{\cos(\beta+\phi)}W, \quad N = -\frac{\cos(\beta-\alpha)}{\cos(\beta+\phi)}W\cos\phi;$$

$$F = -\frac{\cos(\beta-\alpha)}{\cos(\beta+\phi)}W\sin\phi;$$

$$P = -\frac{\sin(\phi+\alpha)}{\cos(\beta+\phi)}W = \frac{\sin\alpha+\mu\cos\alpha}{\mu\sin\beta-\cos\beta}W. \qquad \beta < +90 \text{ and } > +(90-\phi).$$

(9) *Find the force P necessary to just move a cylinder of radius R and weight W up a rough plane inclined at an angle α, by a crow-bar of length l inclined at an angle β.* (For smooth surface see Ex. (3), page 174.)

Ans. The weight W can be resolved into two components R_1 and R_2 making the angles of repose ϕ_1 and ϕ_2 with the normals at the points of contact D_1 and D_2, where ϕ_1 is the angle of repose for the bar and cylinder and ϕ_2 for the plane and cylinder.



We have then $R_1 = \dfrac{\sin(\phi_2+\alpha)}{\sin[(\alpha+\beta)+(\phi_2-\phi_1)]}W.$

The normal pressure at D_1 is then

$$N_1 = R_1\cos\phi_1.$$

If P acts at right angles to the bar, we have by virtual work, for a small displacement due to turning of the bar about A through an indefinitely small angle θ,

$$Pl\theta - N_1 . \overline{AD_1} . \theta = 0, \quad \text{or} \quad P = \frac{N_1 . AD_1}{l}.$$

But $AD_1 = r\tan\dfrac{1}{2}\left(\alpha+\beta\right) = \dfrac{r[1-\cos(\alpha+\beta)]}{\sin(\alpha+\beta)}$. Hence

$$P = \frac{Wr\cos\phi_1\sin(\phi_2+\alpha)[1-\cos(\alpha+\beta)]}{l\sin(\alpha+\beta)\sin[(\alpha+\beta)+(\phi_2-\phi_1)]}.$$

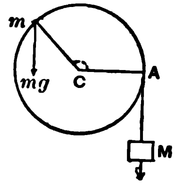
If $\phi_1 = \phi_2$, we have after reduction

$$P = \frac{Wr\cot\phi\sin(\phi+\alpha)}{l[1+\cos(\alpha+\beta)]}.$$

If there is no friction, $\phi = 0$, and we have the same result as in Ex. (3), page 174.

(10) *A particle of mass m rests on a rough cylinder and is held in equilibrium by a string fastened to another particle of mass M, which passes over the cylinder and hangs freely. Determine the position of equilibrium.* (For smooth cylinder see Ex. (4), page 174.)

Ans. From page 202, if the arc of contact $mOA = \alpha$, we have for the friction of the cord



$$F_1 = mg(\epsilon^{\mu\alpha} - 1),$$

where μ is the coefficient of static sliding friction between cord and cylinder and $\epsilon = 2.3026$ = base of Naperian system of logarithms, and g is the acceleration of gravity (page 3.

The normal pressure of m is $mg \sin\alpha$, and the friction of the particle m is

$$F_2 = \mu_2 mg \sin\alpha,$$

where μ_2 is the coefficient of static sliding friction between the particle and cylinder.

The tangential component of mg is $-mg\cos\alpha$. We have then, for equilibrium,

$$-mg\cos\alpha + \mu_2 mg\sin\alpha + mg(\epsilon^{\mu\alpha} - 1) = Mg,$$

or

$$-m(\cos\alpha - \mu_2\sin\alpha) + m(\epsilon^{\mu\alpha} - 1) = M;$$

or, since $\mu_2 = \dfrac{\sin\phi_2}{\cos\phi_2}$,

$$-\cos(\alpha + \phi_2) + (\epsilon^{\mu\alpha} - 1)\cos\phi_2 = \frac{M}{m}\cos\phi_2,$$

from which α can be found. If there is no friction, $\mu = 0$, $\phi_2 = 0$, and

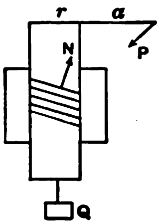
$$\cos\alpha = -\frac{M}{m},$$

which is the same result as in Ex. (4), page 174. If we neglect the friction of the cord, $\mu = 0$ and

$$\cos(\alpha + \phi_2) = -\frac{M}{m}.$$

(11) *Find the conditions for equilibrium for a rough screw.* (For smooth screw see Ex. (5), page 175.)

Ans. Let P be the force applied at the end of the arm a, and let the radius of the screw be r, the pitch p, and the mass supported Q.



If N is the sum of the normal pressures and α the inclination of the thread to the horizontal, we have $N = \dfrac{Q}{\cos\alpha}$ and the friction $F = \mu N = \dfrac{\mu Q}{\cos\alpha}$, where μ is the coefficient of static sliding friction.

If P has a virtual displacement of θ radians, Q is raised a distance $\dfrac{p\theta}{2\pi}$, the distance of the friction is $\dfrac{r\theta}{\cos\alpha}$, and we have by virtual work

$$Pa\theta - \frac{Qp\theta}{2\pi} - \frac{\mu Qr\theta}{\cos^2\alpha} = 0.$$

We have then, since $\dfrac{p}{2\pi r} = \tan\alpha$, $\mu = \tan\phi$,

$$P = \frac{Q}{a}\left(\frac{p}{2\pi} + \frac{\mu r}{\cos^2\alpha}\right) = \frac{Qr}{a}\left(\tan\alpha + \frac{\tan\phi}{\cos^2\alpha}\right).$$

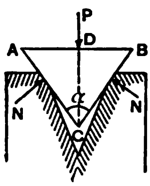
If we neglect friction, we have $\mu = 0$ and $P = \dfrac{Qp}{2\pi a} = \dfrac{Qr\tan\alpha}{a}$, which is the same result as in Ex. (5), page 175.

(12) ***Find the conditions for equilibrium for the differential screw given in Ex. (6), page 175, taking friction into account.***

$$\text{Ans. } P = \frac{Q}{a}\left[\frac{p-p'}{2\pi} + \frac{\mu(r+r')}{\cos^2\alpha}\right]$$

(13) ***Let the force acting normally at the middle of the back AB of a rough isosceles wedge ABC be P, and let the normal pressure on each side be N. Find the conditions for equilibrium.*** (For smooth wedge see Ex. (7), page 176.)

Ans. Let the angle of the wedge at the point C be α. The forces which sustain the wedge in equilibrium are P, the pressures N and the friction F along each face, which acts opposite to the direction in which motion tends to take place.



If μ is the coefficient of static sliding friction, we have $F = \mu N$.

If we put the algebraic sum of the components along the axis DC equal to zero, we have for equilibrium

$$-P + 2N\sin\frac{\alpha}{2} \pm 2\mu N\cos\frac{\alpha}{2} = 0,$$

where the (+) sign is taken for wedge on the point of entering and the (−) sign for wedge on the point of sliding out.

Since $\mu = \frac{\sin\phi}{\cos\phi}$, where ϕ is the angle of repose, we have

$$P = 2N\left(\sin\frac{\alpha}{2} \pm \mu\cos\frac{\alpha}{2}\right) = \frac{2N}{\cos\phi}\sin\left(\frac{\alpha}{2} \pm \phi\right).$$

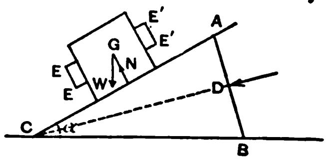
If $P < \frac{2N}{\cos\phi}\sin\left(\frac{\alpha}{2} = \phi\right)$ and $> \frac{2N}{\cos\phi}\sin\left(\frac{\alpha}{2} - \phi\right)$, the wedge is neither on the point of going in or out and we have non-limiting equilibrium. If $\frac{\alpha}{2} = \phi$, there is no force required to prevent the wedge from sliding out. The angle α of the wedge should not then exceed 2ϕ.

If we neglect friction, $\phi = 0$, and we have $P = 2N\sin\frac{\alpha}{2}$.

This is the same result as in Ex. (7), page 176.

(14) *Let a rough isosceles wedge rest with one face BC on a horizontal plane. Let a normal force P act at the middle point of the back. Let the body GHK, whose weight is W, rest upon the face AC and be constrained by guides to move in a normal to AC. Find the conditions for equilibrium.* (For smooth wedge see Ex. (8), page 176.)

Ans. Let N be the normal pressure between the surface AC and the body. Then the friction between the body and wedge is μN, where μ is the coefficient of static sliding friction between the body and wedge. This friction acts opposite to the direction in which motion tends to take place. It is then a pressure upon the guide E or E' according as the wedge is on the point of entering or sliding back. If W is the weight of the body acting at the centre of mass G, then $W\sin\alpha$ is the pressure upon the guide E by reason of the weight. The total pressure upon the guide E is then



$$W\sin\alpha \pm \mu N,$$

according as the wedge enters or slides back. The friction between the body and guide is then

$$\mu_1(W \sin \alpha \pm \mu N),$$

where μ_1 is the coefficient of static sliding friction for body and guide.

We have then for equilibrium of the body

$$N - W \cos \alpha \mp \mu_1 (W \sin \alpha \pm \mu N) = 0,$$

where the upper signs are for wedge on point of entering, and the lower signs for wedge on point of sliding out. Hence

$$N = \frac{W(\cos \alpha \pm \mu_1 \sin \alpha)}{1 - \mu\mu_1}.$$

If we put this value of N in the value for P found in the preceding example, we have

$$P = \frac{2W(\cos \alpha \pm \mu_1 \sin \alpha)}{1 - \mu_1\mu}\left(\sin \frac{\alpha}{2} \pm \mu \cos \frac{\alpha}{2}\right),$$

or, since $\mu_1 = \frac{\sin \phi_1}{\cos \phi_1}$ and $\mu = \frac{\sin \phi}{\cos \phi}$, where ϕ_1 and ϕ are the angles of repose for body and guide, and body and wedge,

$$P = \frac{2W \cos (\alpha \mp \phi_1)}{\cos (\phi_1 + \phi)} \cdot \sin \left(\frac{\alpha}{2} \pm \phi\right).$$

The upper signs are for wedge on the point of entering, the lower signs for wedge on the point of sliding out.

Here again, if $\frac{\alpha}{2} = \phi$, no force P is required to prevent the wedge from sliding out.

If

$$P < \frac{2W \cos (\alpha - \phi_1)}{\cos (\phi_1 + \phi)} \sin \left(\frac{\alpha}{2} + \phi\right) \text{ and } > \frac{2W \cos (\alpha + \phi_1)}{\cos (\phi_1 + \phi)} \sin \left(\frac{\alpha}{2} - \phi\right),$$

we have non-limiting equilibrium and the wedge is not on the point of moving either way. If we neglect friction $\phi = 0$, $\phi_1 = 0$, and $P = 2W \cos \alpha \sin \frac{\alpha}{2}$. This is the same result as in Ex. (8), page 176.

(15) *Solve the case of Ex.* (16), *page* 177, *taking friction into account.*

Ans. $\tan \theta = \frac{a \cot (\alpha_1 + \phi) - b \cot (\alpha_2 - \phi)}{a + b}$,

$$N_1 = \frac{P \sin (\alpha_2 - \phi) \cos \phi}{\sin [(\alpha_1 + \phi) + (\alpha_2 - \phi)]}, \qquad N_2 = \frac{P \sin (\alpha_1 + \phi) \cos \phi}{\sin [(\alpha_1 + \phi) + (\alpha_2 - \phi)]}.$$

(16) *A rod rests with its ends against a rough vertical and horizontal plane. The weight P of the rod acts at its middle point. Find the conditions of equilibrium.*

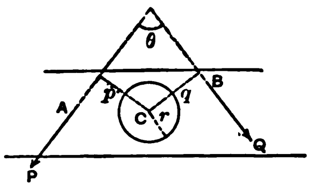
Ans. Let θ be the angle with the horizontal and N_1, N_2 be the normal pressures on the horizontal and vertical planes respectively. Then

$$\tan \theta = \cot 2\phi, \quad N_1 = P \cos^2 \phi, \quad N_2 = P \sin \phi \cos \phi.$$

(17) *A rough lever ACB rests on an axle of radius r and is acted upon by the co-planar forces P and Q applied at the points A and*

B. The forces make the angle θ. Find the relations of P to Q for equilibrium. (For smooth lever see Ex. (1), page 161.)

Ans. The resultant of P and Q is



$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta},$$

the acute value of θ being taken.

We have seen (page 196) that for well-greased axle and small surface of contact we can take, in all cases of axle friction, the friction $F = \mu R$, where μ is the coefficient of static sliding friction.

Let the radius of the axle be r, the lever-arm of P with reference to the centre C of the axle be p, and the lever-arm of Q be q.

We have then in general for equilibrium

$$Pp - Qq \mp \mu Rr,$$

or

$$Pp = Qq \pm \mu r\sqrt{P^2 + Q^2 + 2PQ\cos\theta},$$

where the upper sign is to be taken when rotation in the direction of P just begins, and the lower sign when rotation in the direction of Q just begins.

If the forces P and Q are parallel, $R = P + Q$, and we have

$$P = \frac{q \pm \mu r}{p \mp \mu r}.$$

For all values of P *less than* the first of these values, or

$$P < \frac{q + \mu r}{p - \mu r}Q,$$

and at the same time *greater than* the second, or

$$P > \frac{q - \mu r}{p + \mu r}Q,$$

we have non-limiting equilibrium and the lever is not upon the point of rotating in either direction.

If we neglect friction, $\mu = 0$ and $P = \frac{q}{p}Q$, as in Ex. (1), page 161.

For *partially worn bearing* (page 196) we can put more accurately

$$\sin\phi \textit{ in place of } \phi,$$

where ϕ is the angle of repose.

For *triangular bearing* (page 197) we can put more accurately

$$\frac{\sin 2\phi}{2\cos\alpha} \textit{ in place of } \mu,$$

where α is the half angle of bearing.

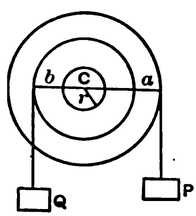
For *new bearing* (page 198) we can put more accurately

$$\frac{\alpha\sin 2\phi}{2\sin\alpha} \textit{ in place of } \mu,$$

where α is the half angle of contact.

(18) *In a wheel and axle the radius of the wheel is a, and of the axle b. Find the conditions for equilibrium, taking into account friction and the rigidity of the rope, when a mass P hung from the*

wheel just balances a mass Q hung from the axle. (Without friction and rigidity see Ex. (2), page 162.)



ANS. We have seen (page 196) that for well-greased axle and small surface of contact we can take in all cases of axle friction the friction $F = \mu R = \mu(P + Q)$, where μ is the coefficient of static sliding friction.

Let the radius of the axle be r, and let t be the thickness of the rope.

Then when P is just about to fall, we have (page 202) for the lever-arm of Q, $\left(1 + \frac{T'}{Q}\right)\left(b + \frac{t}{2}\right)$, and hence for equilibrium

$$-P\left(a + \frac{t}{2}\right) + Q\left(1 + \frac{T'}{Q}\right)\left(b + \frac{t}{2}\right) + \mu r(P + Q) = 0,$$

or

$$P = \frac{\left(b + \frac{t}{2} + \mu r\right)Q + \left(b + \frac{t}{2}\right)T'}{a + \frac{t}{2} - \mu r},$$

where (page 203)

$$\textit{for hemp ropes } T' = \frac{c_1 + c_2 Q}{b + \frac{t}{2}};$$

$$\textit{for wire ropes } T' = c_1 + \frac{c_2 Q}{b + \frac{t}{2}};$$

the values of c_1 and c_2 being given on page 203.

When Q is just about to fall, we have (page 203), for the lever-arm of P, $\left(1 + \frac{T'}{P}\right)\left(a + \frac{t}{2}\right)$, and hence

$$-P\left(1 + \frac{T'}{P}\right)\left(a + \frac{t}{2}\right) + Q\left(b + \frac{t}{2}\right) - \mu r(P + Q) = 0,$$

or

$$P = \frac{\left(b + \frac{t}{2} - \mu r\right)Q - \left(a + \frac{t}{2}\right)T'}{a + \frac{t}{2} + \mu r},$$

where (page 203)

$$\textit{for hemp ropes } T' = \frac{c_1 + c_2 P}{a + \frac{t}{2}};$$

$$\textit{for wire ropes } T' = c_1 + \frac{c_2 P}{a + \frac{t}{2}};$$

the values of c_1 and c_2 being given on page 203.

For values of P less than the first and greater than the second, we have non-limiting equilibrium, and the wheel and axle is not upon the point of rotating in either direction.

If we neglect friction and rigidity, we have $P = \dfrac{b + \frac{t}{2}}{a + \frac{t}{2}} Q$, or, neglecting the thickness of the rope, $P = \dfrac{b}{a} Q$, as in Ex. (2), page 162.

If $b = a$, we have the case of the single pulley.

For *partially worn bearing* (page 196) we can put more accurately

$$\sin \phi \textit{ in place of } \mu,$$

where ϕ is the angle of repose.

For *triangular bearing* (page 197) we can put

$$\frac{\sin 2\phi}{2 \cos \alpha} \textit{ in place of } \mu,$$

where α is the half angle of the bearing.

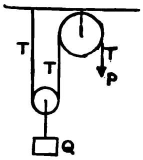
For *new bearing* (page 198) we can put

$$\frac{\alpha \sin 2\phi}{2 \sin \alpha} \textit{ in place of } \mu,$$

where α is the half angle of contact.

(19) *In the single movable pulley find the relation between the force P and the mass Q for equilibrium, taking into account friction and the rigidity of the rope.* (Without friction and rigidity see Ex. (5), page 163.)

Ans. Let r be the radius of the axle of each pulley, a the radius of each pulley, t the thickness of rope, μ the coefficient of static sliding friction, and c_1, c_2 as given on page 203.



For convenience of notation let

$$u = a + \frac{t}{2} + \mu r + c_2, \qquad w = a + \frac{t}{2} - \mu r.$$

Then from the preceding example, making $b = a$, we have, when P is just about to fall, for *hemp ropes*

$$P = \frac{uT_1 + c_1}{w},$$

where T_1 is the tension in the first rope as shown in the figure.

We have in the same way

$$T_1 = \frac{uT_2 + c_1}{w}.$$

We have also $\qquad T_1 + T_2 = Q.$

Eliminating T_1 and T_2, we have

$$P = \frac{u^2 Q + (w + 2u)c_1}{w(w + u)}.$$

In the same way we find when P is on the point of rising

$$P = \frac{(u - 2\mu r - c_2)^2 Q - c_1(w + 2u - 2\mu r - c_2)}{(w + u)(u - 2\mu r)}.$$

For values of P less than the first and greater than the second, we have non-limiting equilibrium and P is not on the point of falling or rising.

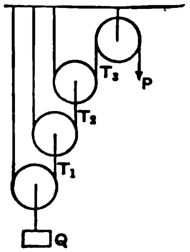
For *wire ropes* we have only to substitute $c_1\left(a + \frac{t}{2}\right)$ in place of c_1.

For partially worn bearing or new bearing we can replace μ by the values given in the preceding example.

If we neglect friction and rigidity, we have $P = \frac{Q}{2}$ as in Ex. (5), page 163.

(20) *In the system of pulleys shown, find the relation between the force P and the mass Q for equilibrium, taking into account friction and rigidity of the rope.* (Without friction and rigidity see Ex. (6), page 163.)

Ans. Let m be the mass of each movable pulley, and n the number of movable pulleys. Let r be the radius of the axle of each pulley, a the radius of each pulley, μ the coefficient of static sliding friction, t the thickness of the rope, and c_1 and c_2 as given on page 203.



For convenience of notation let

$$u = a + \frac{t}{2} + \mu r + c_2; \quad w = a + \frac{t}{2} - \mu r;$$

$$v = u + w = 2a + t + c_2.$$

Then, from the preceding example, we have, when P is just about to fall, for *hemp ropes*

$$T_1 = \frac{u(Q+m)}{v} + \frac{c_1}{v};$$

$$T_2 = \frac{u(T_1+m)}{v} + \frac{c_1}{v};$$

$$T_3 = \frac{u(T_2+m)}{v} + \frac{c_1}{v};$$

and so on. Inserting the values of T_1 and T_2, we have in general

$$T_n = \frac{u^n Q}{v^n} + \frac{(mu + c_1)(u^n - v^n)}{v^n(u - v)}.$$

But from the preceding example we have

$$P = \frac{uT_n}{w} + \frac{c_1}{w}.$$

Hence, since $u - v = -w$,

$$P = \frac{u}{wv^n}\left[u^n Q + \frac{(mu + c_1)(v^n - u^n)}{w}\right] + \frac{c_1}{w}.$$

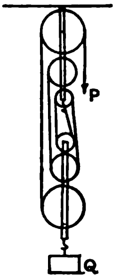
For *wire ropes* we have only to substitute $c_1\left(a + \frac{t}{2}\right)$ in place of c_1.

For partially worn bearing or new bearing we replace μ by the values given in Ex. (18).

If we neglect friction and rigidity, we have $\frac{u}{w} = 1$, $\frac{u}{v} = \frac{1}{2}$, $v = 2\left(a + \frac{t}{2}\right)$, $u = a + \frac{t}{2}$ and $c_1 = 0$, and this reduces to $P = \frac{Q + (2^n - 1)m}{2^n}$, which is the same result as given in Ex. (6), page 163.

(21) *In the system of pulleys shown, find the relation between the force P and the mass Q for equilibrium, taking into account friction and the rigidity of the ropes.* (Without friction and rigidity see Ex. (7), page 164.)

Ans. Let m be the mass of the lower block, and n the number of ropes coming from the lower block. Let r be the radius of the axle of each pulley, μ the coefficient of static sliding friction, t the thickness of the rope, and c_1 and c_2 as given on page 203.



Let a be the *mean radius of the pulleys.*

For convenience of notation let

$$u = a + \frac{t}{2} + \mu r + c_2, \quad w = a + \frac{t}{2} - \mu r.$$

Then we have for *hemp ropes*, when P is about to descend,

$$P = \frac{u^n(u-w)}{w(u^n - w^n)}\left[(Q+m) + \frac{c_1}{u-w}\right] - \frac{c_1}{u-w}.$$

For *wire ropes* we have only to substitute $c_1\left(a + \frac{t}{2}\right)$ in place of c_1.

For partially worn bearing or new bearing, we replace μ by the values given in Ex. (18).

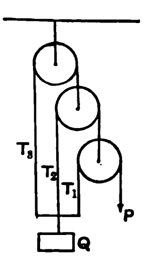
If we neglect friction and rigidity, we have $u = w$ and $c_1 = 0$. The value of P reduces then to $P = \frac{0}{0}$; but if we divide numerator and denominator by $u - w$ and then make $u = w$, we have

$$P = \frac{Q+m}{n},$$

which is the same result as given in Ex. (7), page 164.

(22) *In the system of pulleys shown, find the relation between the force P and the mass Q for equilibrium, taking into account friction and the rigidity of the ropes.* (Without friction and rigidity, see Ex. (8), page 164.)

Ans. Let m be the mass of each pulley and n the number of pulleys. Let r be the radius of the axle of each pulley, μ the coefficient of static sliding friction, t the thickness of the rope and c_1, c_2 as given on page 203.



Let a be the radius of each pulley, and for convenience of notation let

$$u = a + \frac{t}{2} + \mu r + c_2, \quad w = a + \frac{t}{2} - \mu r.$$

Then we have, when P is about to descend, for *hemp ropes*

$$P = \frac{Q + nm - \frac{mu}{w}\left[\left(\frac{w}{u}+1\right)^n - 1\right] + \frac{c_1}{w}\left[\left(\frac{w}{u}+1\right)^n - 1\right]}{\left(\frac{w}{u}+1\right)^n - 1}.$$

For *wire ropes* we have only to substitute $c_1\left(a + \frac{t}{2}\right)$ in place of c_1.

For partially worn bearing or new bearing we replace μ by the values given in Ex. (18).

If we neglect friction and rigidity, we have $u = w$ and $c_1 = 0$, and

$$P = \frac{Q + nm - (2^n - 1)m}{2^n - 1},$$

which is the same result as given in Ex. (8), page 164.

(23) *In the differential pulley of Ex.* (12), *page* 165, *find the relation of P to Q for equilibrium, taking into account friction.*

Ans. Let m be the mass of each pulley, r the radius of each axle, and μ the coefficient of static sliding friction. Since the pulley is worked by a chain, we can disregard rigidity and have only friction to take into account. We have then for P about to descend

$$P = \frac{\left(Q + m\right)\left(\frac{a-b}{2}\right) + 2\mu r(Q + 2m)}{a - 2\mu r}.$$

For partially worn bearing or for new bearing we can replace μ by the values given in Ex. (18). If we neglect friction and the mass of the pulleys, we have $P = \frac{Q(a-b)}{2a}$, which is the same result as in Ex. (12), page 165.

(24) *Solve Ex.* (24), *page* 180, *when the surfaces are rough.*

Ans. Let μ be the coefficient of static sliding friction, and ϕ be the angle of friction. Then, taking the same rotation as in Ex. (24), page 180,

$$a \tan \alpha + \frac{b \sin (\alpha + \phi)}{\cos \alpha \cos \phi} = a \tan (\theta - \phi).$$

$$d = l \cos \alpha - r \cos \theta.$$

(25) *Solve Ex.* (25), *page* 180, *when the surface is rough.*

Ans. Let ϕ be the angle of friction. Then, taking the same notation as in Ex. (25), page 180, we obtain

$$W \cos (\theta + \phi) = \frac{y}{p} H_1 \sin (\theta - \phi).$$

APPLICATIONS OF STATICS.

CHAPTER I.

RETAINING WALLS, DAMS AND EARTH SLOPES.

DEFINITIONS OF PARTS OF A WALL. WEIGHT AND FRICTION OF MASONRY. STABILITY OF A MASONRY JOINT. STABILITY OF A WALL IN GENERAL. LOW GRAVITY DAM. HIGH GRAVITY DAM. ECONOMIC SECTION FOR A HIGH GRAVITY DAM. THE ARCH DAM. THE RETAINING WALL. GRAPHIC AND ANALYTIC DETERMINATION OF THE EARTH PRESSURE ON A RETAINING WALL. COHESION OF EARTH. EQUILIBRIUM OF AN EARTH MASS. EARTH SLOPES AND TERRACES.

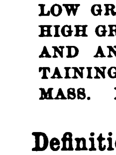
Definitions of Parts of a Wall.—The **face** of a wall is the front surface, or outside surface, or the surface farthest from the pressure. The **back** is the rear surface, or inside surface, or the surface which sustains pressure.

The stone which forms the face is called the **facing**; that which forms the back, the **backing**; that which forms the interior, the **filling**.

A horizontal layer of stone in a wall is called a **course**. If the stones in each layer are of the same thickness, we have **regular courses**; if they are not of the same thickness, we have **irregular** or **random courses**.

The mortar layer between the stones is the **joint**. The horizontal joints are **bed-joints**.

Cut stone or squared masonry is called **ashlar**. Unsquared masonry is called **rubble**.



The inclination of the face or back of a wall, measured by the ratio of its horizontal to its vertical projection, is called the **batter** of the face or back. The batter is then the tangent of the angle which the face or back makes with the vertical. Thus in the figure the batter of the side AD is $\frac{AO}{DO} = \tan\beta$, where β is the **batter angle** or angle of AD with the vertical.

Weight and Friction of Masonry.—We give here a short Table of average values of the coefficient of static sliding friction μ, the corresponding angle of friction or repose ϕ, and the density or mass of a cubic foot δ for different kinds of masonry.

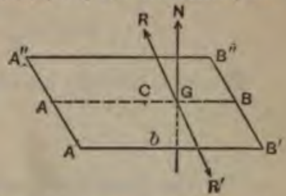
In discussing the stability of walls, the influence of the mortar is neglected, both because of its uncertain character and because the error is on the side of safety. The values given for μ and ϕ are therefore for dry masonry.

We also give in the Table average values of the allowable compressive unit stress C in tons per square foot, taking 2000 lbs. to a ton.

We also give the specific mass (page 10), $\frac{\delta}{\gamma}$, of the materials, where γ is the mass of a cubic foot of water $= 62.5$ lbs.

Kind of Masonry.	Coefficient of Friction. μ	Angle of Friction. ϕ	Density Pounds per Cubic Foot. δ	Specific Mass. $\frac{\delta}{\gamma}$	Allowable Compressive Unit Stress C. Tons per square foot.
Limestone and granite:					
Ashlar masonry.......	0.6	31°	165	2.64	25 to 30
Large mortar rubble...	0.6	31°	150	2.40	10 to 15
Small dry rubble......	0.6	31°	125	2.00	6 to 10
Concrete..................	0.6	31°	150	2.40	12 to 17
Sandstone:					
Ashlar masonry.......	0.6	31°	150	2.40	20 to 25
Large mortar rubble...	0.6	31°	130	2.08	10 to 15
Small dry rubble..... .	0.6	31°	110	1.76	6 to 10
Brickwork.................	0.6	31°	100	1.60	6 to 10

Stability of a Masonry Joint.—Let $A'B'B''A''$ be the area of a joint between two rectangular plane surfaces, as, for instance, between two layers of stone in a masonry structure. Let AB be the line passing through the centre of mass C of the area and the middle points A and B of the opposite sides $A'A''$ and $B'B''$. Let $AB = b$ be the breadth of joint, $A'A'' = l$ the length, ϕ the angle of friction of the dry joint, disregarding the effect of the mortar, $\mu = \tan \phi$ the corresponding coefficient of static sliding friction, C the allowable compressive unit stress, R' the resultant of all the external forces acting at the point G in the line AB.



The values of ϕ, μ and C are given in the Table.

Then we have the following conditions for stability:

1st. The resultant R' of all the external forces must intersect the joint at some point G *within the surface of contact* (page 169); otherwise we have rotation.

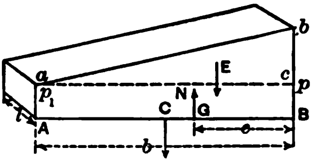
2d. The resultant reaction R of the joint at this point G must be equal and opposite to R' and, if we disregard the effect of the mortar, must make an angle RGN with the normal to the surface AB, *less than the angle of friction or repose* ϕ (page 189); otherwise we have sliding.

It is customary in the discussion of the stability of masonry structures to disregard the effect of the mortar because of its uncertain character and because the error is on the side of safety.

3d. The greatest unit pressure at any point of the joint *must not*

exceed the allowable compressive unit stress C for the materials in contact; otherwise the joint is overloaded.

Determination of this Greatest Unit Pressure.—Let N be the normal component of the resultant reaction R acting at the point G.



Then the least unit pressure p_1 will act along the farthest edge at A, and the greatest unit pressure p will act along the nearest edge at B. If we lay off $Aa = p_1$ and $Bb = p$, the unit pressure at any other point will be given by the ordinate to the straight line ab, and the total load will be represented by the area $ABba$ multiplied by the area of the joint bl.

We have then the mean unit pressure $\frac{p_1 + p}{2}$, and hence the total pressure

$$N = \frac{p_1 + p}{2} . bl, \quad \text{or} \quad p_1 = \frac{2N}{bl} - p. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let $e = BG$ be the "edge distance" or distance of N from the *nearest edge B.*

The entire load area is made up of the rectangular area $ABca$ and the triangular area acb. The load represented by the rectangular area is p_1bl, and its centre of action is at C at a distance $BC = \frac{b}{2}$ from the edge B. The load represented by the triangular area is $\frac{p - p_1}{2} . bl$, and its centre of action is at E at a distance of $\frac{1}{3}b$ from the edge B.

We have then, taking moments about the edge B, for equilibrium,

$$p_1bl \times \frac{b}{2} + \frac{(p - p_1)bl}{2} \times \frac{b}{3} - Ne = 0. \quad . \quad . \quad . \quad (2)$$

From (1) and (2) we obtain for the greatest unit pressure

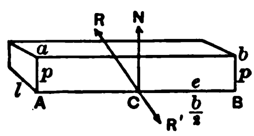
$$p = \frac{2N}{bl}\left(2 - \frac{3e}{b}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and for the least unit pressure

$$p_1 = \frac{2N}{bl}\left(\frac{3e}{b} - 1\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

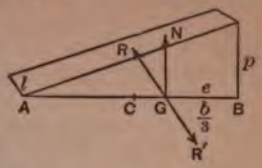
where N is the total normal pressure on the joint acting at a distance e from the nearest edge, l is the length of joint, b the breadth.

In any case, then, we can find the value of p from (3), and this value *must not exceed the allowable compressive unit stress C for the materials in contact,* otherwise the joint is overloaded.

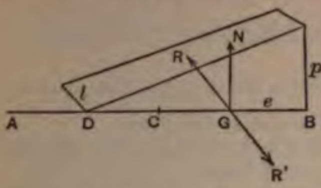


We see from (3) and (4) that when $e = \frac{b}{2}$ we have $p_1 = p = \frac{N}{bl}$. That is, when the resultant R' of all the external forces acts at the centre of mass C of the joint, the load N is uniformly distributed over the entire joint, and the unit pressure at every point is $p = \frac{N}{bl}$.

As e diminishes, p increases and p_1 decreases; and when $e = \frac{b}{3}$, we have $p_1 = 0$ and $p = \frac{2N}{bl}$. That is, when the resultant R' of all the external forces acts at $\frac{b}{3}$ from the nearest edge, the unit pressure at the farthest edge is zero, and the greatest unit pressure at the nearest edge is twice as great as if the load were uniformly distributed over the area of the entire joint bl.



If then e is *less* than $\frac{b}{3}$, the whole joint is not brought into action. The effective area of joint is $3el$, or the distance $BD = 3e$. The portion AD affords no resistance, if we disregard the effect of the mortar, and the greatest unit pressure is $p = \frac{2N}{3el}$, or twice as great as if the load were uniformly distributed over the effective area $3eb$.



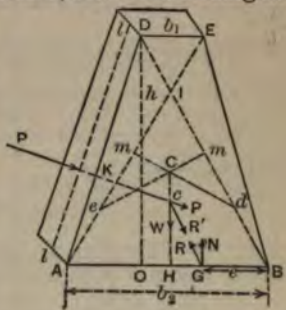
We see then—

4th. That, in order to just bring the entire joint AB into action, the resultant R' of all the external forces must intersect the joint *at the middle third*.

This is called the "*middle third rule*," and in an economically proportioned masonry structure it should be complied with.

Stability of a Wall.—Let $ABDE$ be the section of a wall. We can investigate its stability as follows:

1. **By Graphic Construction.**—Find the centre of mass C of the section (page 22) by drawing the diagonals AE, BD intersecting at I. Lay off along these diagonals $Ae = IE$, and $Bd = ID$, and let m, m be the middle points of AE and BD. Join md and me. The intersection C is the centre of mass.



At the centre of mass thus found let the weight W of the section of wall act. Let b_2 be the bottom base AB, and b_1 the upper base DE, and l the length and h the height DO. Then the volume of the section is $\frac{(b_1 + b_2)hl}{2}$. If δ is the density or mass of a unit of volume of the masonry, we have the weight W in gravitation units,

$$W = \frac{(b_1 + b_2)hl\delta}{2}.$$

(For values of δ see page 229.)

Let P be the resultant pressure upon the wall in gravitation units, acting at the point K and known in magnitude and direction. Since we can consider P as acting at any point in its line of direction, produce it till it meets the line of direction of W at the point c. Let W and P both act at this point c, and find their resultant R'.

Then, as we have just seen in the preceding Article:

1st. The resultant R' must intersect the joint AB at some point G within the base; otherwise we have rotation.

2d. If the joint AB *extends through the wall*, the reaction R of the surface AB at G must be equal and opposite to R' and make an angle RGN with the normal N less than the angle of friction or repose ϕ; otherwise we have sliding. (For values of ϕ see page 229.) For security we should have

$$n \times \text{angle } RGN = \phi,$$

where n is called the **factor of safety for sliding.** In practice n should be at least 2 or even more if shocks are to be apprehended.

The student should note that if the joint AB *does not extend through the wall, no investigation for sliding is necessary.*

3d. The greatest unit pressure must not be greater than the allowable compressive unit stress C for the materials in contact; otherwise the base AB is overloaded. (For values of C see page 229.)

4th. For economic proportions $e = GB$ must be just equal to $\frac{1}{3}b_2$, in which case the entire base AB is just brought into action.

If e is greater or less than $\frac{1}{3}b_2$, the proportions are not economic, but stability exists in any case if condition 3d is fulfilled.

We make then the construction as directed on page 231. If the joint AB extends through the wall, we must have

$$n \times \text{angle } RGN = \phi,$$

where n should be 3 or more if shocks are to be apprehended.

If the joint AB does not extend through the wall, there is no danger of sliding.

If the construction gives $e = \frac{1}{3}b_2$, the proportions are economic, and there is also stability provided that (page 230)

$$\text{for } e = \frac{1}{3}b_2 \qquad p = \frac{2N}{lb_2} \leqq C.$$

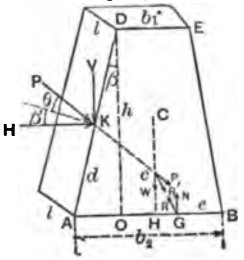
If the construction gives e greater or less than $\frac{1}{3}b_2$, the proportions are not economic, but we still have stability provided that

$$\text{for } e > \frac{1}{3}b_2 \qquad p = \frac{2N}{lb_2}\left(2 - \frac{3e}{b_2}\right) \leqq C,$$

and provided that

$$\text{for } e < \frac{1}{3}b_2 \qquad p = \frac{2N}{3el} \leqq C.$$

2. **By Calculation.**—Let the back of the wall AD make the batter-angle β with the vertical, and the pressure P make the angle θ with the normal to the wall and therefore the angle $(\beta + \theta)$ with the horizontal.



Then the vertical component of P is

$$V = P \sin (\beta + \theta), \quad . \quad . \quad . \quad (1)$$

and the horizontal component of P is

$$H = P \cos (\beta + \theta). \quad . \quad . \quad . \quad (2)$$

If δ is the density or mass of a unit of volume of the masonry, we have for the weight W of the section

$$W = \frac{(b_1 + b_2)lh\delta}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(For values of δ see page 229.)

Hence the normal component N of R is

$$N = W + V. \qquad (4)$$

If μ is the coefficient of static sliding friction for the base AB, we have for limiting equilibrium, if the joint AB extends through the wall, the friction

$$F = \mu N.$$

(For values of μ see page 229.)

In order that the angle RGN shall be less than the angle of friction ϕ, we must have

$$H < F.$$

For security let us put $nH = F$, or

$$nH = \mu(W + V).$$

Then we have

$$n = \frac{\mu(W + V)}{H}, \qquad (I)$$

where V, H and W are given in any case by (1), (2) and (3).

We call n the **factor of safety for sliding**. If n is less than unity, the wall slides. If $n = 1$, we have $H = F$ or limiting equilibrium, and the wall is on the point of sliding. For safety, then, n must be greater than unity, and the greater it is the greater the security. If $n = 2$ or 3, it will take two or three times the given pressure P to make the wall just begin to slide. In practice n should be at least two or even more, if shocks are to be apprehended. *If the joint AB does not extend through the wall, there is no danger of sliding and equation (I) need not be applied.*

Let the distance AK of the point of application of P from A be d. Let $e = GB$ be the distance of the intersection of the resultant R' and the base AB from the edge B of the wall.

Take the point G as the point of moments. Then the lever-arm of the horizontal component H of P is $d \cos \beta$, the lever-arm of the vertical component V of P is $(b_2 - d \sin \beta - e)$, and the lever-arm of the weight W is $(b_2 - \overline{AH} - e)$, where the horizontal distance $\overline{AH}$ of W from A is given (page 22) by

$$s_2 = \overline{AH} = \frac{b_2}{2} - \frac{b_2 + 2b_1}{3(b_1 + b_2)}\left[\frac{b_2 - b_1}{2} - h \tan \beta\right]. \qquad (5)$$

We have then for equilibrium, taking moments about the point G,

$$-Hd \cos \beta + V(b_2 - d \sin \beta - e) + W(b_2 - s_2 - e) = 0,$$

or

$$e = \frac{W(b_2 - s_2) + V(b_2 - d \sin \beta) - Hd \cos \beta}{W + V}, \qquad (II)$$

where V, H and W are given by (1), (2) and (3), and $AH = s_2$ is given by (5).

For economic proportions we should have $e = \frac{1}{3} b_2$. If then we put $e = \frac{1}{3} b_2$ in (II) and solve for b_2, we have

$$b_2 = -B + \sqrt{B^2 + E},$$

where for convenience of notation

$$\left.\begin{aligned} B &= \frac{1}{2}\left[b_1 + \frac{4V}{\delta hl} - h \tan\beta\right]; \\ E &= b_1(b_1 + 2h\tan\beta) + \frac{6d}{\delta hl}(V\sin\beta + H\cos\beta). \end{aligned}\right\} \quad . \; . \quad \text{(III)}$$

Equations (III) give us the length of the lower base $AB = b_2$ for economic proportions, when the entire base AB just comes into action.

If b_2 has this value, we must have for security against overloading (page 230)

$$\text{for } e = \frac{1}{3}b_2 \qquad p = \frac{2(W+V)}{lb_2} \overline{\gtrless} C, \quad . \; . \; . \; . \; . \quad (6)$$

where C is the allowable compressive unit stress as given on page 229.

If b_2 is greater or less than the value given by (III), or if e as given by (2) is greater or less than $\frac{1}{3}b_2$, the proportions are not economic, but we still have stability if the base AB is not overloaded, that is, provided that in the first case (page 230)

$$\text{for } e > \frac{1}{3}b_2 \qquad p = \frac{2(W+V)}{lb_2}\left(2 - \frac{3e}{b_2}\right) \overline{\gtrless} C, \quad . \; . \; . \quad (7)$$

and provided that in the second case

$$\text{for } e < \frac{1}{3}b_2 \qquad p = \frac{2(W+V)}{3el} \overline{\gtrless} C. \quad . \; . \; . \; . \; . \; . \quad (8)$$

It is the custom of some engineers, for the sake of additional security, to neglect the vertical component V of the pressure in equations (I), (II) and (III). In such case we have only to make $V = 0$ in these equations.

Low and High Wall.—If e, as given by equation (II), is less than or equal to $\frac{1}{3}b_2$ and at the same time conditions (8) or (6) are found to be satisfied, so that the base AB is not overloaded, the wall is called a "*low*" wall. In such case b_2 may be made equal to or less than its value as given by (III).

When, however, the wall is so high that, when e is equal to $\frac{1}{3}b_2$, condition (6) cannot be satisfied, it is called a "*high*" wall. In such case b_2 must be greater than its value as given by (III), and e must be greater than $\frac{1}{3}b_2$.

To find the limiting value of b_2 in this case: from condition (7) let

$$\frac{2(W+V)}{lb_2}\left(2 - \frac{3e}{b_2}\right) = C, \quad \text{or} \quad e = \frac{2}{3}b_2 - \frac{Clb_2^2}{6(W+V)}.$$

Let e in equation (II) have this value, and solve for b_2, and we have

$$
\left.
\begin{aligned}
&b_2 = -K + \sqrt{K^2 + L},\\
&\text{where, for convenience of notation,}\\
&K = \frac{1}{C}\left[\frac{V}{l} - \frac{\delta h^2 \tan\beta}{2}\right],\\
&L = \frac{\delta h b_1}{C}(b_1 + 2h\tan\beta) + \frac{6d}{lC}(V\sin\beta + H\cos\beta),
\end{aligned}
\right\} \quad \text{(IV)}
$$

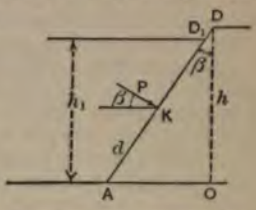
where V and H are given by (1) and (2) If the vertical component V of the pressure is neglected, as is the custom of some engineers for the sake of additional security, we have only to make $V = 0$ in (IV).

Equations (IV) give the least value of b_2 for a "*high*" wall, that is, for a wall so high that when $e = \frac{1}{3}b_2$ the base AB is overloaded.

Low Gravity Dam.—A wall which resists the pressure of water by reason of its weight alone is called a "*gravity dam.*" It is a "*low*" dam if e can be equal to or less than $\frac{1}{3}b_2$, without overloading the base.

The general investigation of the stability of a wall given in the preceding Article applies to any case where the pressure P is known in direction, point of application and magnitude.

Direction of Water Pressure.—It is a well-known principle of Physics that the direction of water pressure upon a submerged surface *is always normal to the surface.*

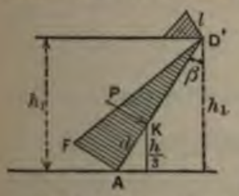


We have then in the formulas of the preceding Article

$$\theta = 0,$$

and the angle of the pressure P with the horizontal is equal to the batter-angle of the back $ADO = \beta$.

Point of Application of Water Pressure.—Moreover, since the pressure at the water level D' is zero and the pressure at any point increases directly as the depth of that point below the water level, the pressure at any point is proportional to the ordinate to a straight line $D'F$, and the resultant pressure P acts at the centre of mass of a triangle $AD'F$, that is, at a distance $AK = d$ equal to $\frac{h_1}{3\cos\beta}$, where h_1 is the depth of water back of the wall.



In the formulas of the preceding Article we have then

$$d = \frac{h_1}{3\cos\beta}.$$

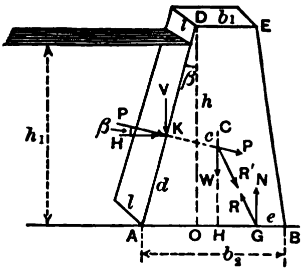
Magnitude of Water Pressure.—It is also a well-known principle of Physics that the *pressure is equal to the weight of a prism of water whose base is the submerged surface and whose height is the distance from the water level to the centre of mass of the submerged surface.*

The submerged surface is $\frac{lh_1}{\cos\beta}$, where l is the length and $\frac{h_1}{2}$ is the distance of the centre of mass of the submerged surface from the water level. Let γ be the density or mass of a unit of volume of water (62.5 lbs. per cubic foot). Then we have for the pressure

$$P = \frac{\gamma l h_1^2}{2\cos\beta}.$$

We have then for the vertical component of P

$$V = \frac{\gamma l h_1^2}{2}\tan\beta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$



and for the horizontal component of P

$$H = \frac{\gamma l h_1^2}{2}. \quad . \quad . \quad . \quad . \quad (2)$$

The weight W of the dam is

$$W = \frac{(b_1 + b_2)lh\delta}{2} = Al\delta, \quad . \quad (3)$$

where A is the area of the cross-section $ABED$.

(For values of δ see page 229.)

If then we substitute $\theta = 0°$, $d = \frac{h_1}{3\cos\beta}$, and the values of V, H and W as given by (1), (2) and (3) in the general formulas of the preceding Article, we obtain the corresponding formulas for a dam sustaining water pressure only. The graphic construction is the same as on page 231.

Ice and Wave Pressure.—A dam, however, has to sustain, in addition to the water pressure on the back, a horizontal pressure at the top surface due to waves or the thrust of ice. We denote this horizontal thrust *per linear foot* of dam, due to waves or ice, by T. For waves we may take $T = 24000$ pounds per linear foot, and for ice $T = 40000$ pounds per linear foot. Since both these do not act together, we have only to consider T for ice in cold climates and T for waves in warm.

Factor of Safety for Sliding.—The normal component N of R is

$$N = W + V, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and the friction is

$$F = \mu N = \mu(W + V),$$

where μ is the coefficient of static sliding friction for the base AB. For values of μ see page 229.

If n is the factor of safety for sliding, we have

$$n(H + T) = F,$$

or

$$\left.\begin{array}{l} n = \dfrac{\mu(W + V)}{H + T}, \\[2ex] \text{or, if } V \text{ is neglected,} \\[2ex] n = \dfrac{\mu W}{H + T}, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{I})$$

where V, W and H are given by (1), (2) and (3). If there are no through joints in the dam, there can be no sliding and equation (I)

need not be applied. If there are through joints, n should be at least 2 or more if shocks are to be apprehended.

Stability and Proportions.—We have for the horizontal distance $\overline{AH} = s_2$ of the centre of mass of the section from A (page 22)

$$\overline{AH} = s_2 = \frac{b_2}{2} - \frac{b_2 + 2b_1}{3(b_1 + b_2)}\left[\frac{b_2 - b_1}{2} - h \tan \beta\right]. \quad . \quad . \quad (5)$$

If we take moments about the point G (figure, page 236), we have as on page 233, taking the ice-thrust T into account,

$$- Hd \cos \beta - Th_1 + V(b_2 - d \sin \beta - e) + W(b_2 - s_2 - e) = 0,$$

or, substituting $d = \dfrac{h_1}{3 \cos \beta}$ and the values of V and W,

$$\left.\begin{aligned} e &= \frac{A(b_2 - s_2) + \dfrac{\gamma h_1^2}{2\delta} \tan \beta\left(b_2 - \dfrac{h_1}{3} \tan \beta\right) - \dfrac{\gamma h_1^2}{6\delta} - \dfrac{Th_1}{\delta}}{A + \dfrac{\gamma h_1^2}{2\delta} \tan \beta}; \\ &\text{or, if we neglect } V, \\ e &= \frac{A(b_2 - s_2) - \dfrac{\gamma h_1^2}{6\delta} - \dfrac{Th_1}{\delta}}{A}, \end{aligned}\right\} . \quad \text{(II)}$$

where A is given by (3), and s_2 by (5). Equation (II) gives the point at which the resultant cuts the base when the ice-thrust acts.

For economic proportions we should have $e = \frac{1}{3} b_2$ *when the ice- or wave-thrust T does not act.* Putting, then, $e = \frac{1}{3} b_2$ in (II) and neglecting T and solving for b_2, we have

$$\left.\begin{aligned} b_2 &= - B + \sqrt{B^2 + E}, \\ &\text{where} \\ B &= \frac{1}{2}\left[b_1 + \frac{2\gamma h_1^2 \tan \beta}{\delta h} - h \tan \beta\right]; \\ E &= b_1(b_1 + 2h \tan \beta) + \frac{\gamma h_1^3}{\delta h}(1 + \tan^2 \beta); \\ &\text{or, if } V \text{ is neglected,} \\ B &= \frac{1}{2}(b_1 - h \tan \beta);\ E = b_1(b_1 + 2h \tan \beta) + \frac{\gamma h_1^3}{\delta h}. \end{aligned}\right\} . \quad \text{(III)}$$

Equations (III) give the lower base $b_2 = AB$ for economic proportions, that is, when $e = \frac{1}{3} b_2$, or the whole base AB just comes into action *when there is no ice- or wave-thrust T.* If b_2 has this value, we must have for security against overloading (page 230)

$$\text{when } e = \frac{1}{3} b_2 \qquad p = \frac{2A\delta + \gamma h_1^2 \tan \beta}{b_2} \lesseqgtr C, \quad . \quad . \quad . \quad (6)$$

where C is the allowable compressive unit stress as given page 229.

If b_2 is taken greater or less than the value given by (III), the value of e given by (II) when T is neglected will be greater or less

than $\frac{1}{3}b_2$ and the proportions are not economic. But we still have stability if the base is not overloaded, that is, if

$$\text{when } e > \frac{1}{3}b_2 \quad p = \frac{2A\delta + \gamma h_1^2 \tan\beta}{b_2}\left(2 - \frac{3e}{b_2}\right) \overline{\overline{<}} C, \quad . \quad . \quad (7)$$

and if

$$\text{when } e < \frac{1}{3}b_2 \quad p = \frac{2A\delta + \gamma h_1^2 \tan\beta}{3e} \overline{\overline{<}} C. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

But now, *when the ice-thrust T acts*, e is given by (II); and in order that the base may not be overloaded, this value of e must satisfy condition (8). If it does not, the ice- or wave-thrust T causes the base to be overloaded. Substituting then the value of e from (II) in (8), and the value of s_2 from (5), and neglecting V, and making $\beta = 0$, we have, since $A = \frac{(b_1 + b_2)h}{2}$,

$$b_2 = -\frac{(C - \delta h)b_1}{(2C - \delta h)} + \sqrt{\frac{b_1^2(C - \delta h)^2}{(2C - \delta h)^2} + \frac{hb_1^2(C + \delta h) + Ch_1(\gamma h_1^2 + 6T)}{\delta h(2C - \delta h)}}. \quad (\text{III}')$$

Equation (III′) gives *the least value of b_2 consistent with safety when the ice- or wave-thrust T acts, for vertical back.* For the sake of security and simplicity we take the same limiting value when the back is not vertical. If then condition (8) is not satisfied when we take for e its value from (II), we cannot have economic proportions, but must take b_2 equal to or greater than the value given by (III′).

It is the custom of some engineers, for the sake of additional security, to neglect the vertical component V of the pressure in equations (I), (II) and (III). We have therefore given these equations for both cases.

When the dam is empty, equation (5) gives the intersection of the weight with the base. In this case

$$\text{when } s_2 = \frac{1}{3}b_2 \quad \text{we must have } p = \frac{2A\delta}{b_2} \overline{\overline{<}} C;$$

$$\text{" } \quad s_2 > \frac{1}{3}b_2 \quad \text{" } \quad \text{" } \quad \text{" } \quad p = \frac{2A\delta}{b_2}\left(2 - \frac{3s_2}{b_2}\right) \overline{\overline{<}} C;$$

$$\text{" } \quad s_2 < \frac{1}{3}b_2 \quad \text{" } \quad \text{" } \quad \text{" } \quad p = \frac{2A\delta}{3s_2} \overline{\overline{<}} C.$$

When the back is vertical, $\beta = 0$ and (5) becomes

$$s_2 = \frac{1}{3}b_2 + \frac{b_1^2}{3(b_1 + b_2)}.$$

That is, s_2 *is always greater than* $\frac{1}{3}b_2$ *for vertical back.*

We can put B in equation (III) in the form

$$B = \frac{1}{2}b_1 + \frac{\gamma h \tan\beta}{2\delta}\left(\frac{2h_1^2}{h^2} - \frac{\delta b}{\gamma}\right).$$

We see from the Table page 229 that the specific mass $\frac{\delta}{\gamma}$ is greater than 2 for all materials except brickwork and small dry rubble. We can never have h_1 or the depth of water greater than

h or the height of wall. Hence for all materials except brick and small dry rubble the term in the parenthesis is minus, and even for the last two materials it is minus if h_1 is not more than $\frac{9}{10}h$. In general, then, B *increases* and E *decreases* as the angle β decreases.

In the value for b_2, then, the magnitude of B increases more rapidly than $\sqrt{B^2 + E}$, and b_2 has its least value when $\beta = 0$.

Hence the most economical section of dam is that which has the back vertical.

High Gravity Dam.—If e as given by equation (II), page 237, is less than or equal to $\frac{1}{3}b_2$, and at the same time conditions (8) or (6), are satisfied, so that the base AB is not overloaded, the dam is "*low*." In such case b_2 may be made equal to or less than its value as given by (III), provided it is greater than the least value given by (III′).

When, however, the dam is so high that when $e = \frac{1}{3}b_2$ condition (6) cannot be satisfied, it is called "*high*." In such case b_2 must be greater than its value as given by (III), and e must be greater than $\frac{1}{3}b_2$.

To find the limiting value of b_2 in this case: From condition (7), page 238, let

$$\frac{2A\delta + \gamma h_1^2 \tan\beta}{b_2}\left(2 - \frac{3e}{b_2}\right) = C, \quad \text{or} \quad e = \frac{2}{3}b_2 - \frac{Cb_2^2}{6A\delta + 3\gamma h_1^2 \tan\beta}.$$

Let e in equation (II), page 237, have this value and solve for b_2, and we have (page 235)

$$\left.\begin{aligned}
&b_2 = -K + \sqrt{K^2 + L},\\
&\text{where}\\
&K = \frac{\tan\beta}{2C}(\gamma h_1^2 - \delta h^2),\\
&\text{or, if } V \text{ is neglected,}\\
&K = -\frac{\delta h^2 \tan\beta}{2C},\\
&\text{and}\\
&L = \frac{\delta h b_1}{C}(b_1 + 2h\tan\beta) + \frac{\gamma h_1^3}{C\cos^2\beta} + \frac{6h_1}{C}T,\\
&\text{or, if } V \text{ is neglected,}\\
&L = \frac{\delta h b_1}{C}(b_1 + 2h\tan\beta) + \frac{\gamma h_1^3}{C} + \frac{6h_1}{C}T;
\end{aligned}\right\} \quad . \quad . \quad \text{(IV)}$$

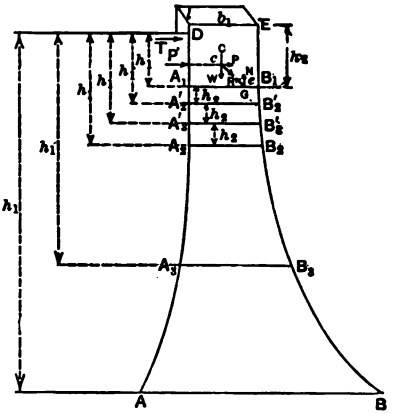
where C is the allowable compressive unit stress as given page 229, h_1 is the depth of water, h the height of section, γ the density or mass of a unit of volume of water, δ the density or mass of a unit of volume of masonry, β the batter-angle of back, b_1 the breadth at top of section and b_2 at bottom.

Equations (IV) give the least value of b_2 for a "*high*" dam, that is, so high that when $e = \frac{1}{3}b_2$ the base AB is overloaded.

Since it is the custom of some engineers, for the sake of additional security, to neglect the vertical component V of the pressure, we have given these equations for both cases.

Economic Section for High Gravity Dam.—We have seen, page 239, that the economic section for alow dam has the back vertical.

First Sub-section.—Let $DE = b_1$ be the top base. The economic section of the first sub-section A_1B_1ED should then be a rectangle for a distance h_2 such that $e = B_1G$ shall be just equal to $\frac{1}{3}b_1$, so that the entire joint A_1B_1 may act, provided this joint is not overloaded.



We find the height h_2 of this rectangular portion as follows:

If h_1 is the depth of water above A_1B_1, the horizontal pressure is $P = \frac{\gamma l h_1^2}{2}$, where γ is the density or mass of a unit of volume of water, and l is the length of dam considered. This pressure P acts at $\frac{1}{3}h_1$ above A_1B_1. The weight of the sub-section is $W_1 = \delta l h_2 b_1 = A_1 l \delta$, where A_1 is the area and δ is the density or mass of a unit of volume of masonry. It acts at $\frac{1}{2}b_1$ from B_1.

Taking moments about G and neglecting the ice-thrust T, we have

$$W_1\left(\frac{b_1}{2} - e\right) - \frac{Ph_1}{3} = 0,$$

or, when $e = \frac{1}{3}b_1$, inserting the values of W_1 and P_1,

$$\delta h_2 b_1^2 = \gamma h_1^3.$$

Let a be the distance of the water level below the top of the dam, then $h_1 = h_2 - a$. Substituting this, we have

$$\left.\begin{array}{l} \delta b_1^2 h_2 = \gamma (h_2 - a)^3, \\ \text{or for the extreme case of water level with top of dam,} \\ a = 0 \quad \text{and} \quad h_2 = b_1\sqrt{\frac{\delta}{\gamma}}, \end{array}\right\} \quad . \quad . \quad \text{(I)}$$

where $\frac{\delta}{\gamma}$ is the specific mass (page 10) of the masonry as given page 229.

The same result is obtained from equation (III), page 237, by making $\beta = 0$, $b_2 = b_1$, and $h_1 = h_2$.

Equation (I) gives the height of the first rectangular sub-section AB_1ED, provided the joint A_1B_1 is not overloaded.

If there are no through joints, there is no danger of sliding.

Top Thickness.—If now we consider the ice-thrust T as acting and take moments about G, we have

$$W_1\left(\frac{b_1}{2} - e\right) - \frac{Ph_1}{3} - Tlh_1 = 0,$$

or, substituting the values of W_1 and P,

$$e = \frac{b_1}{2} - \frac{\gamma h_1^3}{6\delta b_1 h_2} - \frac{Th_1}{\delta b_1 h_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We obtain the same result from equation (II), page 237, by making $\beta = 0$, $b_2 = b_1$, $s_2 = \frac{b_1}{2}$.

For the extreme case of water level with top of dam, $h_1 = h_2$; and if we substitute the value of h_2 from (I), we have

$$e = \frac{b_1}{3} - \frac{T}{\delta b_1}.$$

But in order that A_1B_1 may not be overloaded, we must have

$$\frac{2A_1\delta}{3e} = C, \quad \text{or} \quad e = \frac{2\delta h_2 b_1}{3C},$$

where C is the allowable unit stress of compression. We have then

$$\frac{2\delta h_2 b_1}{3C} = \frac{b_1}{3} - \frac{T}{\delta b_1},$$

or, substituting the value of h_2 from (I),

$$\frac{2\delta\sqrt{\frac{\delta}{\gamma}}}{C} . b_1^3 - b_1^2 = -\frac{3T}{\delta}.$$

A high dam would be built of ashlar masonry, and we have from page 229 the average values $\delta = 150$, $\frac{\delta}{\gamma} = 2.5$, $C = 50000$. Taking $T = 40000$, we have for the average value of b_1 which allows (I) to be fulfilled without overloading, when water is level with top of dam,

$$0.0096b_1^3 - b_1^2 = -800, \quad \text{or} \quad b_1 = \text{about } 35 \text{ ft.}$$

When the top base b_1, then, is *about* 35 *ft. or over*, we can run the first rectangular sub-section A_1B_1ED down for the distance given by (I) without danger of overloading when the ice-thrust acts.

Local and practical considerations must control the choice of top base b_1. But if it is taken less than about 35 ft., e is given by (1) and we must have

$$\frac{2A_1\delta}{3e} = C, \quad \text{or} \quad 2\delta b_1 h_2 = 3eC,$$

or, substituting the value of e from (1) and putting $h_1 = h_2 - a$, where a is the depth of water below the top,

$$2\delta b_1 h_2^2 = \frac{3}{2} b_1 C h_2 - \frac{\gamma C (h_2 - a)^3}{2\delta b_1} - \frac{3CT(h_2 - a)}{\delta b_1}. \quad . \quad . \quad (\text{I}')$$

From (I′) we can find the height h_2 of the rectangular sub-section A_1B_1ED when b_1 is less than 35 ft. and the ice-thrust T acts.

We find then the first rectangular sub-section A_1B_1ED from (I) if b_1 is greater than 35 ft., and from (I′) if b_1 is less than 35 ft., and the joint A_1B_1 will not be overloaded when the ice-thrust acts.

Second Sub-section.—Below A_1B_1 we still continue the back vertical, but b_1 must now increase so that for any joint $b_2 = A_2'B_2'$, e shall be equal to $\frac{1}{3}b_2$, and the joint shall not be overloaded when the ice-thrust T acts.

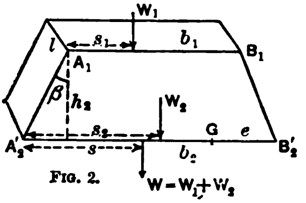


FIG. 2.

Let $A_1B_1B_2'A_2'$ be any section in general below A_1B_1 the height of which h_2 is so small that it may be regarded as a trapezoid. Let β be the batter-angle of the back, W_1 the resultant weight *of all the masonry above* A_1B_1 acting at the distance s_1 from A, W_2 the weight of the section acting at the distance s_2 from A_2', W the resultant $W_1 + W_2$ of these two acting at the distance s from A_2'. Then, taking moments about A_2', we have

$$s = \frac{W_1(s_1 + h_2 \tan\beta) + W_2 s_2}{W_1 + W_2} = \frac{A_1(s_1 + h_2 \tan\beta) + A_2 s_2}{A_1 + A_2}, \quad \text{(II)}$$

where A_1, A_2 are the areas of the sections above A_1B_1 and the section $A_1B_1B_2A_2$, so that $A_1 l\delta = W_1$, $A_2 l\delta = W_2$.

We have then

$$A_2 = \frac{(b_1 + b_2)h_2}{2}, \quad W_2 = \frac{(b_1 + b_2)h_2 l\delta}{2},$$

and, from page 22,

$$s_2 = \frac{b_2}{2} - \frac{b_2 + 2b_1}{3(b_2 + b_1)}\left[\frac{b_2 - b_1}{2} - h_2 \tan\beta\right]. \quad . \quad . \quad (2)$$

Let P be the horizontal component of the water pressure on the *entire back above* $A_2'B_2'$, and h_1 the height of water level above $A_2'B_2'$. Then $P = \frac{\gamma l h_1^2}{2}$, acting at a distance $\frac{h_1}{3}$ above $A_2'B_2'$. We have also the ice-thrust T acting at the distance h_1 above $A_2'B_2'$.

Let the resultant of P, T and W cut the base at the distance $GB_2' = e$ from B_2', Fig. 2. Then, neglecting, for the sake of security and simplicity, the vertical component of the water pressure, we have

$$(W_1 + W_2)(b_2 - s - e) - \frac{\gamma l h_1^3}{6} - Th_1 l = 0;$$

hence

$$e = b_2 - s - \frac{\gamma h_1 \quad + 6Th}{6\delta(A_1 + A_2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

[If in (3) we make $A_1 = 0$, the whole section above $A_2'B_2'$ is a trapezoid and we have the same value for e as from equation (II), page 237, when $\beta = 0$, $s_2 = s$, and $A = A_2$.]

For economic proportions we should have $e = \frac{1}{3}b_2$ *when the ice- or wave-thrust T does not act.* Making, then, $\beta = 0$ in (2) and (II),

substituting the corresponding values of s_2 and s in (II) and (3), and making $e = \frac{1}{3} b_2$ and $T = 0$, we obtain

$$\left.\begin{array}{l} b_2 = -B + \sqrt{B^2 + E}, \\ \text{where} \\ B = \dfrac{2A_1}{h_2} + \dfrac{b_1}{2}, \quad E = \dfrac{6A_1 s_1}{h_2} + \dfrac{\gamma h_1^2}{\delta h_2} + b_1^2. \end{array}\right\} \quad . \quad . \quad \text{(III)}$$

[Here again, if we make $A_1 = 0$, the whole section above $A_2'B_2$ is a trapezoid and we have the same value for b_2 as from equation (III), page 237, when $\beta = 0$, $h = h_2$.]

Equations (III) give the lower base $b_2 = A_2'B_2'$ for economic proportions *when there is no ice- or wave-thrust T.* If then we assume any section $A_1B_1B_2'A_2'$, Fig. 1, page 240, of small depth h_2, we can find by (III) its base $b_2 = A_2'B_2'$, since for this section $s_1 = \frac{1}{2} b_1$. We can then find A_2 and then e, from (3), *when the ice- or wave-thrust T acts.*

This value of e must satisfy the condition

$$\frac{2(A_1 + A_2)\delta}{3e} \overset{=}{<} C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If it does not, the ice- or wave-thrust T causes $A_2'B_2'$ to be overloaded. We have then, taking for the extreme case

$$\frac{2(A_1 + A_2)\delta}{3e} = C, \quad \text{or} \quad 3eC = 2(A_1 + A_2)\delta,$$

and putting for s_2 its value from (2) when $\beta = 0$, and for s its value from (II) when $\beta = 0$, and then from (3) the corresponding value for e, by solving for b_2,

$$\left.\begin{array}{l} b_2 = -B_1 + \sqrt{B_1^2 + E_1}, \\ \text{where} \\ B_1 = \dfrac{A_1(3C - 2\delta h_2)}{h_2(2C - \delta h_2)} + \dfrac{(C - \delta h_2)b_1}{(2C - \delta h_2)}; \\ E_1 = \dfrac{2A_1(2A_1\delta + 2\delta b_1 h_2 + 3Cs_1)}{h_2(2C - \delta h_2)} \\ \qquad + \dfrac{\delta h_2 b_1^2(C + \delta h_2) + Ch_1(\gamma h_1^2 + 6T)}{\delta h_2(2C - \delta h_2)}. \end{array}\right\} \quad . \quad \text{(III')}$$

[This reduces to equation (III'), page 238, when $A_1 = 0$ and $h_2 = h$.]

Equation (III') gives *the least value of b_2 consistent with safety when the ice- or wave-thrust T acts.* If then condition (4) is not satisfied when we take for e its value from (3), we must take for b_2 its value as given by (III').

In either case, whether b_2 is given by (III) or by (III'), we can find s from (II).

This value of s *is the new s_1 for the next section* $A_2'B_2'B_3'A_3'$, Fig. 1, page 240, of small depth h_3. The value of b_2 just found *is the new b_1* for this section. From (III) or (III') we then find b_2 for this section, then s from (II), which is the new s_1 for the next section.

Thus by successive applications of (III) or (III′) and (II) we find successive thicknesses A_2B_2', $A_2'B_2'$, etc., Fig. 1, page 240.

We thus determine the economic section until we arrive at a section A_2B_2, Fig. 1, page 240, for which equation (II) gives us $s = \frac{1}{3}b_2$. When this section is reached equations (III) or (III′) no longer apply, because if the vertical back were continued farther, the resultant pressure *for reservoir empty* would fall outside the middle third, making s less than $\frac{1}{3}b_2$.

We thus determine the lower limit A_2B_2, Fig. 1, page 240, of the second sub-section.

Third Sub-section.—Below this limit we must batter both front and back, so that both e and s shall always be $\frac{1}{3}b_2$ and the joint shall not be overloaded when the ice- or wave-thrust T acts.

If then in (3) we make $s = \frac{1}{3}b_2$ and $e = \frac{1}{3}b_2$ and neglect T, we obtain

$$b_2 = -\left(\frac{A_1}{h_2} + \frac{b_1}{2}\right) + \sqrt{\left(\frac{A_1}{h_2} + \frac{b_1}{2}\right)^2 + \frac{\gamma h_1^3}{\delta h_2}}, \quad . \quad . \quad . \quad \text{(IV)}$$

where b_1 is the top and b_2 the bottom base of any trapezoid of small height h_2, and A_1 the area of all the section above the top base of that trapezoid and h_1 the depth of water above the bottom base of that trapezoid. We can then find the area A_2 of this trapezoid, and then from (3) we can find e when the ice-thrust acts. This value of e must satisfy the condition

$$\frac{2(A_1 + A_2)\delta}{3e} \overline{\overline{<}} C.$$

If it does not, the ice-thrust T causes the base b_2 as given by (IV) to be overloaded. We have then for the least value of b_2 consistent with safety to use (III′) instead of (IV). In either case we can find s_2 from (2), and then from (II), putting $s = \frac{1}{3}b_2$ and solving for $\tan\beta$, we have for the back batter

$$\tan\beta = \frac{A_1\left(\frac{b_2}{3} - s_1\right) - \frac{h_2 b_1^2}{6}}{h_2\left(A_1 + \frac{1}{3}A_2 + \frac{1}{6}h_2 b_1\right)}. \quad . \quad . \quad . \quad . \quad . \quad \text{(V)}$$

We can thus determine by successive applications of (IV) or (III′) and (V) the economic section, until we arrive at a section $b_2 = A_3B_3$, Fig. 1, page 240, for which

$$\frac{2(A_1 + A_2)\delta}{b_2} = C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We thus determine the limit A_3B_3, Fig. 1, page 240, of the third sub-section.

Fourth Sub-section.—Below this limit we must have both s and e greater than $\frac{1}{3}b_2$ and such that (page 230)

$$\frac{2(A_1 + A_2)\delta}{b_2}\left(2 - \frac{3e}{b_2}\right) = C \quad \text{and} \quad \frac{2(A_1 + A_2)\delta}{b_2}\left(2 - \frac{3s}{b_2}\right) = C.$$

Hence

$$e = s = \frac{2}{3}b_2 - \frac{Cb_2^2}{6\delta(A_1 + A_2)}.$$

Substituting these values of e and s in (3) and neglecting T, we obtain

$$b_2 = \frac{\delta(2A_1 + h_2b_1)}{2(2C - \delta h_2)} + \sqrt{\frac{\delta^2(2A_1 + h_2b_1)^2}{4(2C - \delta h_2)^2} + \frac{\gamma h_1^2}{2C - \delta h_2}}. \quad . \quad \text{(VI)}$$

Equation (VI) gives the base b_2 for each successive trapezoid below A_2B_2, Fig. 1, page 240. From (3) we find e when the ice-thrust T acts. This value of e must satisfy the condition

$$\frac{2(A_1 + A_2)\delta}{3e} \overline{\gtrless} C.$$

If it does not, the ice-thrust T causes the base b_2 as given by (VI) to be overloaded. We have then for the least value of b_2 consistent with safety to use (III') instead of (VI). In either case we find the back batter from (V).

Arch Dam.—When the dam is made in the form of an arch so that it supports the water pressure back of it wholly by virtue of its action as an arch, it is called an **arch dam.**

The water pressure upon the back of the dam is always normal to the surface, and the pressure upon a given area is always the same at the same depth.

Let aaa, Fig. 1, be the centre line of a horizontal cross-section of the dam, one foot in height. Let P_1 and P_2 be the equal normal pressures upon the equal portions $a'a'$, $a'a'$, and H the horizontal pressure at the crown.

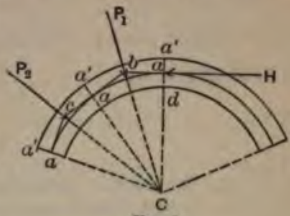
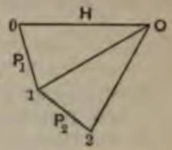


FIG. 1.

In Fig. 2, lay off H from O to 0 horizontally, and let $O\,0$ represent the magnitude of H. Then lay off 01 and 12 parallel and equal in magnitude to P_1 and P_2, and draw the rays $O\,1$, $O\,2$.

FIG. 2.



In Fig. 1, let H act at a, and prolong its direction till it meets P_1 at b. From b draw bc parallel to $O\,1$ till it meets P_2 at c. From c draw ca parallel to $O\,2$.

Then (page 146) $abca$, Fig. 1, is the equilibrium polygon. We have by similar triangles

$$P_1 : H :: cb : bC \text{ or } cC; \quad \therefore \frac{P_1}{cb} = \frac{H}{cC}.$$

The same holds true no matter how many equal portions $a'a'$ we take. But as we increase the number of portions, the polygon approaches a curve. For an indefinitely great number of portions we have for the curve of equilibrium $\frac{P_1}{cb} = p =$ unit pressure and $cC = r =$ radius of curvature. Hence

$$p = \frac{H}{r}, \quad \text{or} \quad r = \frac{H}{p}.$$

But H and p are constant and therefore r is constant. Hence *the curve of equilibrium is a circle.*

If then we make the dam circular in cross-section, *the curve of equilibrium will coincide with the centre line and the horizontal pressure H at the crown acts at the centre line and is equal to*

$$H = rp. \qquad (1)$$

Also, since in Fig. 2 the force polygon 0 1 2 becomes a circle of radius H when the segments of the arch are indefinitely great in number, and since any ray, as 0 1 in Fig. 2, gives the stress in the corresponding segment cb, Fig. 1, of the equilibrium polygon (page 146), it is evident *that the pressure at every point of the centre line is tangent to the centre line at that point and equal to H.*

If then C is the allowable compressive stress per square foot, we have for the area A of the cross-section

$$A = \frac{H}{C} = \frac{rp}{C}.$$

If h_1 is the depth of any point below the water level, we have the water pressure per square foot at that point equal to γh_1, where γ is the mass of a cubic foot of water, or 62.5 lbs. If T is the ice-thrust per foot of length, and h is the height of dam, we have the ice-thrust pressure per square foot of surface of the dam equal to $\frac{T}{h}$.

For an area of one square foot at a depth h_1, then, the total pressure per foot p is numerically equal to $\gamma h_1 + \frac{T}{h}$, and the thickness is given by

$$t = \frac{\gamma r h_1 + \frac{rT}{h}}{C}. \qquad (2)$$

From (2) we can find the thickness of the dam at any point at a depth h_1 below the water level.

If $h_1 = 0$ in (2), we have for the thickness at the water level, or the top thickness b_1, for ice pressure

$$b_1 = \frac{rT}{Ch}. \qquad (3)$$

The choice of top thickness b_1 must in general be determined by local and practical considerations.

If we make $t = b_1 =$ the top thickness in (2), we have for the distance h_2 below the water level for which the cross-section of the dam is a rectangle

$$h_2 = \frac{Cv_1}{\gamma r} - \frac{T}{\gamma h}. \qquad (4)$$

Below this limit the thickness must increase with the depth h_1 according to (2); above it, the thickness is constant and equal to b_1. We should not take h_1 in (2), then, less than h_2 as given by (4).

The arch dam requires far less masonry than the gravity dam. But the pressure on the arch stones increases with the span and with the depth, and so does the thickness. When the thickness becomes great we cannot be sure that each arch stone will take its own share of the pressure. The distribution of the pressure over the cross-section is then uncertain. For such reasons the arch dam is most suitable for short and low dams. It is also manifestly unwise to make the stability of a dam depend wholly upon its action as an arch, except under the most favorable conditions as to rigid

side hills for abutments and the most unfavorable conditions as to cost of masonry.

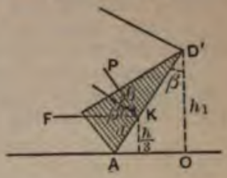
Although it is not, then, generally wise to make the stability of dam depend wholly upon its action as an arch, it is well to make a a gravity dam curved so that the arch action may give additional security.

There are but two dams of the pure arch type in existence: the Zola Dam in the city of Aix in France, and the Bear Valley Dam in the San Bernardino Mts., Southern California. The first is of rubble masonry, height 120 ft., radius 158 ft., thickness at top 19 feet, at base 42 feet. The Bear Valley Dam is of granite, height 64 feet, radius 300 ft., thickness at top 3.16 ft., at base 20 ft.

Retaining Wall.—A wall designed to resist the pressure of earth back of it is called a **retaining wall.**

The general investigation of the stability of a wall given on page 231 applies to any case where the pressure P is known in direction, point of application and magnitude.

Point of Application of P.—In treating retaining walls, it is customary to neglect the cohesion of the earth. We therefore consider the pressure as zero at the earth level and increasing for any point of the back of the wall, directly as the depth of that point below the earth level. The pressure at any point is then proportional to the ordinate to a straight line $D'F$, and the resultant pressure P acts, just as in the case of water pressure, at the centre of mass of the triangle $AD'F$, so that the distance $AK = d$ is one third of AD', or $d = \frac{h_1}{3 \cos \beta}$,



where h_1 is the distance $D'O$ of the earth surface above A, and β is the batter-angle of the back.

But unlike water pressure, the earth pressure is not normal to the wall, but makes an angle θ with the normal.

Also the magnitude of P is not the same as for water.

We have therefore to determine the magnitude and direction of the earth pressure P. We can then investigate the stability precisely as on page 231.

Magnitude and Direction of P—**Graphic Determination.**—Let abc, Fig. 1, be any small prism, and let $+p_1$ be the normal pressure per unit of area upon the faces ac and bc at right angles, the (+) sign indicating direction up and to the right.

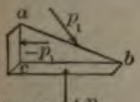


FIG. 2.

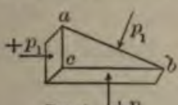


FIG. 1.

Then if there is equilibrium, the pressure per unit of area upon the third face ab *is also normal and equal to* p_1.

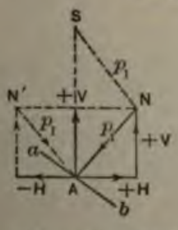


FIG. 3.

For if we multiply the area of the face ac, Fig. 1, by $+p_1$, we have the total horizontal force $+H$, and if we multiply the area of the face bc, Fig. 1, by $+p_1$, we have the total vertical force $+V$. If we lay these forces off in Fig. 3, from A to H, so that $AH = +H$, and from H to N, so that $HN = +V$, the resultant for equilibrium is NA. The line NA in Fig. 3 then gives the magnitude and direc-

tion of the total pressure on the third face ab, Fig. 1, which balances $+p_1 . \overline{ac} = +H$ on the face ac and $+p_1 . \overline{bc} = +V$ on the face bc.

We have then H and V, Fig. 3, perpendicular to the faces ac and bc, Fig. 1, and also

$$\overline{ac} : \overline{bc} :: H : V.$$

Hence the triangles abc, Fig. 1, and NAH, Fig. 3, are similar and NA *is perpendicular* to the face ab.

Also, we have

$$NA = \sqrt{p_1^2 . \overline{ac}^2 + p_1^2 . \overline{bc}^2} = p_1 \sqrt{\overline{ac}^2 + \overline{bc}^2} = p_1 . \overline{ab},$$

or the normal unit pressure p_1 on the face $\overline{ab}$ *is the same for equilibrium as that on the other two faces.*

Suppose now the normal unit pressure p_1 on the face ac, Fig. 2, *to be reversed in direction*, so that it is $-p_1$. We have then the total pressure on the face bc equal to $+p_1 . \overline{bc} = +V$ the same as before, and the total pressure on the face ac equal to $-p_1 . \overline{ac} = -H$, or the same as before in magnitude but *opposite in direction*. If we lay these forces off in Fig. 3, from A to H and H to N', the resultant for equilibrium is $N'A$. It is evident that the magnitude of $N'A$ *is the same as before*, but its direction makes the angle $N'AV$ on the other side of AV equal to the angle NAV in the first case.

If then in Fig. 3 we lay off AN equal to p_1 and with N as a centre and NA as radius describe an arc of a circle intersecting the vertical AV at the point S, then the line SN will give the magni-

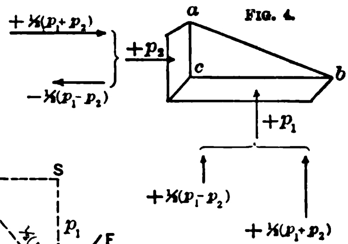


FIG. 4.

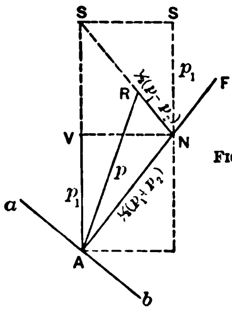


FIG. 5.

tude and direction of the unit pressure p_1 on the face ab in the second case of Fig. 2. The angle ASN is then equal to the angle SAN.

Now suppose that the normal pressures per unit of area on the two faces ac and bc, Fig. 4, are *unequal* and are $+p_2$ and $+p_1$ respectively.

We can divide the normal unit pressure $+p_1$ on the face bc into two parts, one equal to $+\frac{1}{2}(p_1 + p_2)$ and the other equal to $+\frac{1}{2}(p_1 - p_2)$, as indicated in Fig. 4. Similarly, we can divide the

normal unit pressure $+p_2$ on the face ac into two parts, one equal to $+\frac{1}{2}(p_1+p_2)$ and the other equal to $-\frac{1}{2}(p_1-p_2)$.

Then, as we have just proved, the unit pressure *normal to the face ab* which balances $+\frac{1}{2}(p_1+p_2)$ on the face bc and $+\frac{1}{2}(p_1+p_2)$ on the face ac is the same, or NA, Fig. 5, laid off normal to ab, where $NA=\frac{1}{2}(p_1+p_2)$.

Also, as we have proved, the unit pressure on the face ab which balances $+\frac{1}{2}(p_1-p_2)$ on the face bc and $-\frac{1}{2}(p_1-p_2)$ on the face ac is the same, but it makes an angle ASN with the vertical AV equal to SAN. If, then, we lay off, in Fig. 5, AN equal to $\frac{1}{2}(p_1+p_2)$ normal to ab, and with N as a centre and NA as a radius describe an arc of a circle intersecting the vertical AV at the point S, then SN will give the direction of $\frac{1}{2}(p_1-p_2)$ acting on the face ab. Hence if we lay off along this line $NR=\frac{1}{2}(p_1-p_2)$ and join RA, the line RA *will give the magnitude and direction of the resultant unit pressure p on the face ab when the normal unit pressures p_1 and p_2 on the faces bc and ac are unequal.*

Suppose now the faces ac and bc, Fig. 4, to remain invariable in direction, and the normal unit pressures p_2 and p_1 on these faces to remain constant, but let the third face ab vary its inclination with the horizontal. Then the magnitudes of $AN=\frac{1}{2}(p_1+p_2)$ and of $NR=\frac{1}{2}(p_1-p_2)$ in Fig. 5 remain unchanged, but their directions will change as the face ab changes its inclination. It is evident that the greatest possible value of the angle NAR which the resultant unit pressure $p=RA$ on the face ab makes with the normal to that face will be when NR is perpendicular to AR, or when the angle ARN is 90°. *In the case of earth this greatest possible angle is the angle of friction or repose ϕ_1 for earth on earth.*

Also when the angle ARN is 90° and the angle RAN is ϕ_1, the angle SNF of p_1 with the normal AN is equal to $45°+\frac{\phi_1}{2}$.

Let then, in Fig. 6, ab be the surface of a prism of earth, and $AR=p$ be the magnitude and direction of the unit pressure. Draw AN normal to the surface ab, and AR' making the angle of friction ϕ_1 with the normal AN. We can then find by trial a point N in the normal AF, such that if we take N as a centre and NR as a radius, the arc RR' will be just tangent to AR'. When this point N is thus found by trial, the distance AN will be $\frac{1}{2}(p_1+p_2)$, and $NR=NR'$ will be $\frac{1}{2}(p_1-p_2)$. Also, as seen from Fig. 5, if we bisect the angle RNF by the line NS, we obtain the direction NS of p_1, since the angle RNF, Fig. 5, is twice the angle of NA with p_1 or AV.

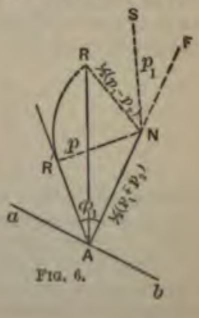
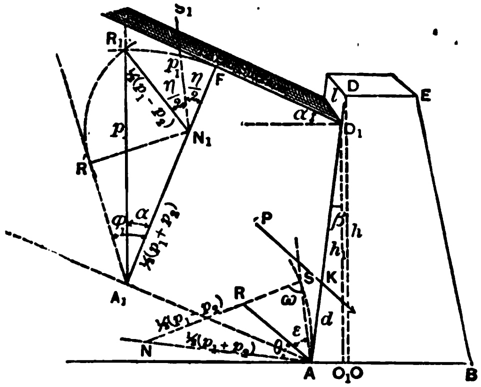


FIG. 6.

Application to the Retaining Wall.—The application of these principles to the retaining wall is obvious.

Let AD be the back of the wall, and D_1FI the earth surface making the angle α with the horizontal. Pass a plane through the



foot of the wall A parallel to the earth surface. The pressure upon every square foot of this plane, as $\overline{ab}$, is *vertical* and equal to the weight of a column of earth of *vertical* height $\overline{A_1I}$ and cross-section $ab \,.\, \cos\alpha$.

If γ_1 is the mass of a cubic foot of earth, then we have

$$\gamma_1 \,.\, \overline{A_1I} \,.\, \overline{ab} \,.\, \cos\alpha$$

for the mass of this column. But $\overline{A_1I}\cos\alpha = \overline{A_1F}$, hence the mass of this column is $\gamma_1 \,.\, \overline{A_1F} \,.\, \overline{ab}$.

If then we draw AF perpendicular to the earth surface and revolve A_1F about A_1 as centre to the vertical A_1R_1, and take the area of ab as one square foot, the distance A_1R_1 in feet will be numerically the same as the number of cubic feet of earth resting upon a square foot ab, and we have for the vertical pressure p per square foot in pounds

$$p = \gamma_1 \,.\, \overline{A_1R_1},$$

where γ_1 is the mass in pounds of a cubic foot of earth, and A_1R_1 is measured in feet.

Then, as in Fig. 6, draw A_1R' making with the normal A_1F to ab the angle $R'A_1F$ equal to the angle of friction or repose ϕ_1 for earth on earth. Find by trial a point N_1 on the normal A_1F, such that the arc of a circle with N_1 as a centre passes through R_1 and is tangent to A_1R'. Then

$$\frac{1}{2}(p_1 + p_2) = \gamma_1 \,.\, \overline{N_1A_1};$$

$$\frac{1}{2}(p_1 - p_2) = \gamma_1 \,.\, \overline{N_1R_1},$$

where γ_1 is the mass in pounds of a cubic foot of earth, and N_1A_1, N_1R_1 are measured in feet. Bisect the angle R_1N_1F by the line N_1S_1. Then the line N_1S_1 gives the direction of p_1 (Fig. 5).

Now lay off at the foot of the wall A (which may be considered as identical with A_1 in the figure) the distance $NA = N_1A_1$ in a direction normal to the back of the wall AD at A. Draw the line

AS parallel to N_1S_1 or the direction of p_1 already found. Then with N as a centre and NA as radius describe an arc of a circle intersecting AS at S, and lay off along NS the distance $NR = N_1R_1$. Then, as in Fig. 5, RA represents the magnitude and direction of the pressure on a square foot at the foot of the wall. Thus, if γ_1 is the mass in pounds of a cubic foot of earth and we measure RA in feet, the pressure per square foot at the foot A of the wall is given in magnitude by

$$\gamma_1 . \overline{RA},$$

and its direction is the direction of RA.

Since the pressure is zero at the top D_1 and greatest at the foot A, and varies for any point directly as the distance of that point from D_1, the average pressure is $\frac{1}{2}\gamma_1 . \overline{RA}$. The total pressure P in pounds is then for a wall one foot in length numerically equal to $\frac{1}{2}\gamma_1 . \overline{RA} . \overline{D'A}$, or if the length of the wall is l,

$$P = \frac{1}{2}\gamma_1 . \overline{RA} . \overline{D'A} . l,$$

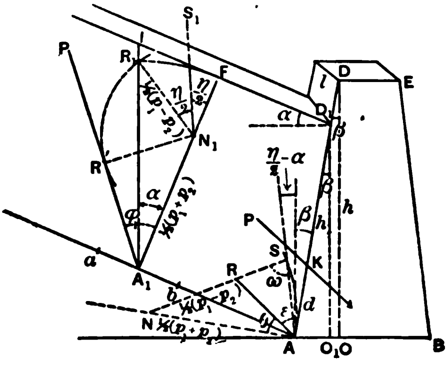
where γ_1 is the mass of a cubic foot of earth, and $\overline{RA}$, $\overline{D_1A}$ and l are taken in feet.

This pressure P acts at a point K at a distance d from the foot of the wall A equal to $d = AK = \frac{1}{3}AD_1$, and is parallel in direction to RA already found.

We thus find by a simple graphic construction, in any given case, the magnitude, direction and point of application of the earth pressure P on the back of the wall. The stability of the wall can then be investigated as directed on page 231.

Analytic Determination of Earth Pressure on a Retaining Wall.—From the graphic construction just given, we can easily derive the corresponding formulas for the magnitude and direction of the earth pressure P.

Notation.—Let $h_1 = D_1O_1$ be the height of the earth surface at D_1 above the base AB of the wall; the angle of the earth surface with the horizontal is α; the batter-angle of the back of the wall



with the vertical is β; the earth pressure P makes the angle θ with the normal to the back of the wall; the angle $R'A_1N_1 = \phi_1$ is the

angle of friction or repose for earth on earth; the angle $R_1N_1F = \eta$, and the angles $R_1N_1S_1 = FN_1S_1 = \frac{\eta}{2}$; the angle $RAS = \epsilon$; the angle $RSA = \omega$—all as indicated in the figure. Finally, γ_1 is the mass of a cubic foot of earth.

Then by the graphic construction we have

$$\frac{1}{2}(p_1 + p_2) \sin \phi_1 = \frac{1}{2}(p_1 - p_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We have also by our notation

$$AD_1 = \frac{h_1}{\cos \beta}, \cdot \ A_1F = AD_1 \cos(\alpha - \beta) = \frac{h_1}{\cos \beta} \cos(\alpha - \beta);$$

and since by construction $A_1R_1 = A_1F$, we have from the figure

$$\frac{1}{2}(p_1 - p_2) \sin \eta = \frac{\gamma_1 h_1}{\cos \beta} \cdot \cos(\alpha - \beta) \sin \alpha. \quad . \quad . \quad . \quad (2)$$

We have also from the figure

$$\left.\begin{aligned}\left[\frac{1}{2}(p_1 + p_2) + \frac{1}{2}(p_1 - p_2) \cos \eta\right]^2 + \left[\frac{1}{2}(p_1 - p_2) \sin \eta\right]^2 \\ = \left[\frac{\gamma_1 h_1}{\cos \beta} \cos(\alpha - \beta)\right]^2,\end{aligned}\right\} \quad . \quad (3)$$

and also

$$\frac{1}{2}(p_1 + p_2) + \frac{1}{2}(p_1 - p_2) \cos \eta = \frac{\gamma_1 h_1}{\cos \beta} \cos(\alpha - \beta) \cos \alpha. \quad . \quad (4)$$

From (1), (2) and (3), eliminating $\frac{1}{2}(p_1 + p_2)$ and $\frac{1}{2}(p_1 - p_2)$, we obtain

$$\cos \eta = -\frac{\sin^2 \alpha}{\sin \phi_1} + \sqrt{\left(1 - \sin^2 \alpha\right)\left(1 - \frac{\sin^2 \alpha}{\sin^2 \phi_1}\right)}. \quad . \quad . \quad (I)$$

We have also directly from the figure ω = angle NAS, or

$$\omega = 90 - \beta - \frac{\eta}{2} + \alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (II)$$

From (2) and (1) we have

$$p_1 = \frac{\gamma_1 h_1 \cos(\alpha - \beta) \sin \alpha (1 + \sin \phi_1)}{\cos \beta \sin \phi_1 \sin \eta}; \quad . \quad . \quad . \quad . \quad (5)$$

$$p_2 = \frac{\gamma_1 h_1 \cos(\alpha - \beta) \sin \alpha (1 - \sin \phi_1)}{\cos \beta \sin \phi_1 \sin \eta}. \quad . \quad . \quad . \quad . \quad (6)$$

We have also from the figure

$$\tan \epsilon = \frac{\overline{RS} \sin \omega}{\overline{AS} - \overline{RS} \cdot \cos \omega}.$$

But $\gamma_1 . \overline{RS} = p_2$, and $\gamma_1 . \overline{AS} = (p_1 + p_2) \cos \omega$. Therefore

$$\tan \epsilon = \frac{p_2 \sin \omega}{p_1 \cos \omega}.$$

Substituting the values of p_1 and p_2 from (5) and (6), we have

$$\tan\epsilon = \frac{1-\sin\phi_1}{1+\sin\phi_1}\tan\omega = \tan^2\left(45^\circ - \frac{\phi_1}{2}\right)\tan\omega. \quad . \quad . \quad \text{(III)}$$

We have also directly from the figure

$$\theta = \omega - \epsilon. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IV)}$$

Also

$$\gamma_1 . \overline{RA} = \sqrt{p_2^2 \sin^2\omega + -(\gamma_1 . \overline{AS} - p_2\cos\omega)^2}$$
$$= \sqrt{p_2^2\sin^2\omega + p_1^2\cos^2\omega},$$

or, substituting the values of p_1 and p_2 from (5) and (6), we have for the earth pressure P,

$$P = \frac{1}{2}\gamma_1 . \overline{RA} . \overline{AD}_1 . l = \frac{lh_1}{\cos\beta} . \frac{1}{2}\gamma_1 . \overline{RA},$$

or

$$P = \frac{\gamma_1 l h_1^2 \cos(\alpha-\beta)\sin\alpha}{2\cos^2\beta\sin\phi_1\sin\eta}\sqrt{(1+\sin\phi_1)^2 - 4\sin\phi_1\sin^2\omega}. \quad \text{(V)}$$

From (1) and (4) we obtain

$$p_1 = \frac{\gamma_1 h_1\cos(\alpha-\beta)\cos\alpha(1+\sin\phi_1)}{\cos\beta(1+\sin\phi_1\cos\eta)}.$$

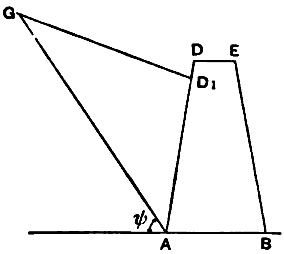
Comparing this with (5), we have

$$\frac{\sin\alpha}{\sin\phi_1\sin\eta} = \frac{\cos\alpha}{1+\sin\phi_1\cos\eta}. \quad . \quad . \quad . \quad . \quad . \quad \text{(7)}$$

We can make this substitution in equation (V) and thus obtain an equivalent expression for P which can be used when α is zero, viz.,

$$P = \frac{\gamma_1 l h_1^2\cos(\alpha-\beta)\cos\alpha}{2\cos^2\beta(1+\sin\phi_1\cos\eta)}\sqrt{(1+\sin\phi_1)^2 - 4\sin\phi_1\sin^2\omega}. \quad \text{(VI)}$$

Surface of Rupture.—If there were no wall and the earth had no cohesion, a prism of earth AD_1G would tend to slide off along a plane AG which would make with the horizontal the angle of repose ϕ_1. But on account of the wall this plane AG makes with the horizontal an angle ψ greater than ϕ_1.



This angle ψ we call the **angle of rupture**, the plane AG is the **plane of rupture**, and the prism AD_1G which thus tends to separate along AG and force the wall is the **prism of rupture**.

If in the figure, page 251, p_1 remains unchanged in direction and magnitude while ab is revolved about A_1 until the pressure upon ab makes with the normal to ab the angle ϕ_1, then this new position of ab gives the inclination of the plane of rupture. But for this new position p_1 makes (page 249) the angle $45 + \frac{\phi_1}{2}$ with the normal. The normal A_1N_1, and hence the plane ab, has then been revolved through the angle $45 + \frac{\phi_1}{2} - \frac{\eta}{2}$.

The angle which the plane of rupture AG makes with the horizontal, or the angle of rupture, is then

$$\psi = 45 + \frac{\phi_1}{2} - \frac{\eta}{2} + \alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(VII)}$$

General Method.—We have then in any case the following method:

1st. Find η from (I).
2d. Find ω from (II).
3d. Find ϵ from (III).
4th. Find θ from (IV).
The angle θ gives the inclination of the pressure with the normal to the back of the wall.
5th. Find P from (V) or (VI).

Then if desired we can find the angle of rupture from (VII).

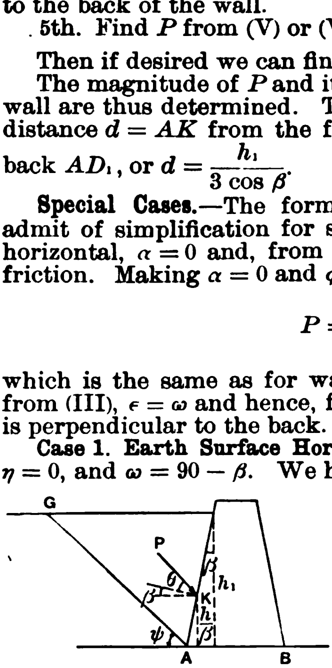
The magnitude of P and its inclination θ with the normal to the wall are thus determined. The point of application K of P is at a distance $d = AK$ from the foot of the wall equal to one third the back AD_1, or $d = \dfrac{h_1}{3 \cos \beta}$.

Special Cases.—The formulas just deduced are general and admit of simplification for special cases. If the earth surface is horizontal, $\alpha = 0$ and, from (I), $\eta = 0$. If ϕ_1 is zero, there is no friction. Making $\alpha = 0$ and $\phi_1 = 0$, we have, from (VI),

$$P = \frac{\gamma_1 l h_1^2}{2 \cos \beta},$$

which is the same as for water pressure (page 236). In this case, from (III), $\epsilon = \omega$ and hence, from (IV), $\theta = 0$, or the water pressure is perpendicular to the back. We have then $\psi = 45^\circ$ for water.

Case 1. Earth Surface Horizontal.—In this case $\alpha = 0$ and hence $\eta = 0$, and $\omega = 90 - \beta$. We have then, from (III),



$$\tan \epsilon = \tan^2\left(45 - \frac{\phi_1}{2}\right) \operatorname{cotan} \beta. \quad . \qquad (8)$$

Then from (IV)

$$\theta = 90 - \beta - \epsilon, \quad . \quad . \qquad (9)$$

and from (VI)

$$P = \frac{\gamma_1 l h_1^2}{2} \sqrt{\frac{1}{\cos^2 \beta} - \frac{4 \sin \phi_1}{(1 + \sin \phi_1)^2}}. \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

From (VII) the surface of rupture AG makes with the horizontal the angle

$$\psi = 45^\circ + \frac{\phi_1}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

Case 2. Earth Surface Horizontal—Back Vertical.—In this case $\alpha = 0$ and $\beta = 0$. Hence $\eta = 0$, $\omega = 90^\circ$ and, from (8), $\epsilon = 90$ and, from (9), $\theta = 0$. The pressure is then *perpendicular to the back or horizontal.* From (10), making $\beta = 0$ and reducing,

$$P = \frac{\gamma_1 l h_1^2}{2} \tan^2\left(45 - \frac{\phi_1}{2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

The surface of rupture makes as before the angle ψ with the horizontal given by

$$\psi = 45° + \frac{\phi_1}{2}.$$

Case 3. Earth Surface Horizontal.—$\beta = 90 - \psi$. In this case $\alpha = 0$, hence $\eta = 0$ and $\psi = 45° + \frac{\phi_1}{2}$. If we make $\beta = 90 - \psi = 45° - \frac{\phi_1}{2}$, we have $\omega = 45° + \frac{\phi_1}{2}$, $\epsilon = 45 - \frac{\phi_1}{2}$ and

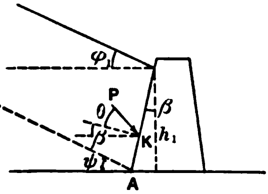
$$\theta = \phi,$$

or the pressure *makes the angle of friction with the normal.*

In this case,

$$P = \frac{\gamma_1 l h_1^2 \cos^2\left(45 + \frac{\phi_1}{2}\right)}{\cos\phi_1 \cos\left(45 - \frac{\phi_1}{2}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Case 4. Earth Surface Inclined at the Angle of Repose.—In this case $\alpha = \phi_1$. Hence



$$\eta = 90 + \phi_1,\ \omega = 45° - \beta + \frac{\phi_1}{2},\ \psi = \phi_1,$$

$\tan\epsilon =$

$$\tan^2\left(45° - \frac{\phi_1}{2}\right) \tan\left(45° - \beta + \frac{\phi_1}{2}\right). \quad (14)$$

$$\theta = 45° - \beta + \frac{\phi_1}{2} - \epsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$P = \frac{\gamma_1 l h_1^2 \cos(\phi_1 - \beta)}{2\cos^2\beta \cos\phi_1} \sqrt{(1 + \sin\phi_1)^2 - 4\sin\phi_1 \sin^2\left(45 - \beta + \frac{\phi_1}{2}\right)}. \quad (16)$$

Case 5. Earth Surface Inclined at the Angle of Repose—Back Vertical.—In this case, $\alpha = \phi_1$, $\beta = 0$, $\eta = 90 + \phi_1$, $\omega = 45° + \frac{\phi_1}{2}$, $\psi = \phi_1$, $\epsilon = 45 - \frac{\phi}{2}$, and hence

$$\theta = \phi_1,$$

or the pressure makes the angle of friction with the normal.

From (16),

$$P = \frac{\gamma_1 l h_1^2}{2} \sqrt{(1 + \sin\phi_1)^2 - 4\sin\phi_1 \sin^2\left(45° + \frac{\phi_1}{2}\right)}. \quad . \quad . \quad (17)$$

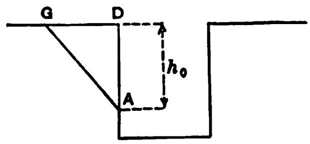
Cohesion of Earth.—**Adhesion** is that resistance to motion which takes place when two different surfaces are in contact. If the surfaces are of the same kind, it is called **cohesion**. It is found by experiment that adhesion or cohesion is directly proportional to the area of contact, varies with the nature of the surfaces in contact, and is independent of the pressure.

It is given then by

$$cA,$$

where A is the area of contact and c is the **coefficient of cohesion** or

adhesion, depending upon the nature of the material. The unit of c is then 1 pound per square foot.



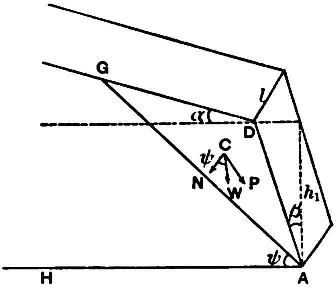
If a trench with vertical sides, of considerable length as compared to its width, is dug in the earth. as shown in the figure, with a transverse trench at each end, so that lateral cohesion may not prevent rupture, after a few days it will be observed to have caved in along some plane as AG. Let the depth AD be h_0.

Then, as we shall see in the next Article, the coefficient of cohesion of the earth is given by

$$c = \frac{\gamma_1 h_0(1 - \sin \phi_1)}{4 \cos \phi_1},$$

where ϕ_1 is the angle of friction or repose, and γ_1 is the mass of a cubic foot of the earth.

Equilibrium of a Mass of Earth. — Let $ADGH$ be a mass of earth, the batter-angle of the face AD being β.



If there were no cohesion, a prism of earth ADG would tend to slide off along a plane AG which would make with the horizontal the angle of repose ϕ_1. But if there is cohesion, this plane, which we have called the **plane of rupture**, will make an angle with the horizontal greater than ϕ_1, which we call the **angle of rupture**.

Let the angle of rupture or the angle of the plane of rupture AG with the horizontal be ψ, the angle of the earth surface DG with the horizontal be α, the length of the mass be l, and the weight of the prism ADG be W.

The weight W acting at the centre of mass C can be resolved into a force N normal to the surface of rupture AG and a force P parallel to the surface.

We have then

$$P = W \sin \psi, \qquad N = W \cos \psi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The force P tends to cause sliding. This force is resisted by the friction and the cohesion. The friction is $\mu_1 N$, where $\mu_1 = \tan \phi_1$ is the coefficient of static sliding friction of the earth, and the cohesion is $cl\,.\,\overline{AG}$, where c is the coefficient of cohesion and $l\,.\,\overline{AG}$ is the area of contact.

We have then for equilibrium

$$P - \mu_1 N - cl\,.\,\overline{AG} = 0, \quad \text{or} \quad P - \mu_1 N = cl\,.\,\overline{AG},$$

or

$$\frac{P - \mu_1 N}{l\,.\,\overline{AG}} = c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now for any plane which makes an angle with the horizontal greater or less than ψ there will be no sliding, and for that plane

$P-\mu_1 N$ will be *less than* $cl.\overline{AG}$, or $\dfrac{P-\mu_1 N}{l.\overline{AG}}$ will be less than c. For the plane of rupture, then, we must have

$$\frac{P-\mu_1 N}{l.\overline{AG}} = \text{a maximum.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let the vertical height of the mass be h_1. Then $\overline{AD} = \dfrac{h_1}{\cos\beta}$, and the weight W of the prism ADG in gravitation units is

$$W = \gamma_1 l . \frac{\overline{AD}}{2} . \overline{AG} . \sin[90 - (\psi + \beta)] = \frac{\gamma_1 l h_1 . \overline{AG}\cos(\psi+\beta)}{2\cos\beta}. \quad (4)$$

Insert this value of W in (1) and the corresponding values of P and N in (3), and we have, since $\mu_1 = \tan\phi_1$,

$$\frac{\gamma_1 h_1 \cos(\psi+\beta)\sin(\psi-\phi_1)}{2\cos\beta\cos\phi_1} = c = \text{a maximum.} \quad . \quad . \quad (5)$$

Angle of Rupture.—Equation (5) is a maximum when

$$\cos(\psi+\beta) = \sin(\psi-\phi_1) = \cos[90-(\psi-\phi_1)],$$

or when

$$\psi + \beta = 90 - \psi + \phi_1,$$

or when

$$\psi = 45 - \frac{\beta}{2} + \frac{\phi_1}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Equation (6) gives then the angle of rupture or the angle which the plane of rupture AG makes with the horizontal.

Coefficient of Cohesion.—If we insert this value of ψ in (5), we obtain

$$\gamma_1 h_1 \sin\left[45 - \frac{1}{2}(\phi_1+\beta)\right]\cos\left[45 + \frac{1}{2}(\phi_1+\beta)\right] = 2c\cos\phi_1\cos\beta,$$

or

$$\gamma_1 h_1[1-\sin(\phi_1+\beta)] = 4c\cos\phi_1\cos\beta. \quad . \quad . \quad . \quad . \quad (7)$$

Now when AD is vertical; $\beta = 0$, and if we denote h_1 in this case by h_0, we have, from (7),

$$c = \frac{\gamma_1 h_0(1-\sin\phi_1)}{4\cos\phi_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This is the value of the coefficient of cohesion given in the preceding Article, where h_0 is found by experiment.

Stability of Slope.—If we substitute the value of c from (8) in (7), we have

$$h_1[1-\sin(\phi_1+\beta)] = h_0(1-\sin\phi)\cos\beta,$$

or

$$h_1 = \frac{h_0(1-\sin\phi_1)\cos\beta}{1-\sin(\phi_1+\beta)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

which is the equation of condition between h_1 and β.

From (6) and (9) we see that the angle of rupture and the relation between h_1 and β are independent of the inclination α of the earth surface with the horizontal.

Equation (9) gives the limiting height h_1 when sliding is about to begin. Let n be the *factor of safety*, so that if n is 2 or 3 the safe height taken can be two or three times as great before sliding begins. Then we have for the safe height

$$h_1 = \frac{h_0(1 - \sin \phi_1) \cos \beta}{n[1 - \sin (\phi_1 + \beta)]}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Equation (10) is then the equation of stability of slope for a factor of safety n, and gives the safe height of slope for any given batter-angle β.

Angle of Stability.—If h_1 is given and the corresponding batter-angle β is required, we can write (10) in the form

$$\frac{1 - \sin (\phi_1 + \beta)}{\cos \beta} = \frac{h_0(1 - \sin \beta_1)}{nh_1} = a,$$

where the second member, being a known quantity, is denoted by a. If we develop the numerator in the first member and substitute for $\sin \beta$ and $\cos \beta$ their values in terms of $\tan \frac{1}{2}\beta$, viz.,

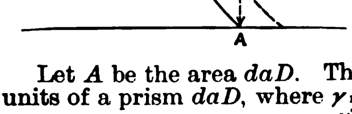
$$\sin \beta = \frac{2 \tan \frac{1}{2}\beta}{1 + \tan^2 \frac{1}{2}\beta}, \qquad \cos \beta = \frac{1 - \tan^2 \frac{1}{2}\beta}{1 + \tan^2 \frac{1}{2}\beta},$$

we obtain a quadratic whose solution gives

$$\tan \frac{1}{2}\beta = \frac{1}{1 + a + \sin \phi_1}[\cos \phi - \sqrt{a(a + 2 \sin \phi)}]. \quad . \quad (11)$$

Equation (11) gives the safe batter-angle β for a factor of safety n when the height h_1 is given.

Curve of Slope.—Let a be any point of the slope DaA, whose vertical distance below D is $da = y$, and let $\overline{aG}$ be the plane of rupture at the point a, making the angle ψ with the horizontal.



Then the prism DGa of weight W tends to slide down along aG and is prevented by friction and cohesion. Let N and P be the components of W normal and parallel to aG. Then if n is the factor of safety and μ is the coefficient of static sliding friction, we have

$$n(P - \mu_1 N) - cl \,.\, \overline{aG} = 0. \quad (12)$$

Let A be the area daD. Then $\gamma_1 lA$ is the weight in gravitation units of a prism daD, where γ_1 is the mass of a unit of volume of the earth. The area daG is $\frac{y^2 \cot \psi}{2}$, and the weight in gravitation units of the prism daG is $\frac{\gamma_1 ly^2 \cot \psi}{2}$. Hence the weight in gravitation units of the prism DaG is

$$W = \gamma_1 l\left(\frac{y^2 \cot \psi}{2} - A\right).$$

If we insert this value of W in the expressions for P and N, equations (1), and then substitute in (12), we obtain, since $aG = \frac{y}{\sin\psi}$,

$$n\gamma_1 l\left(\frac{y^2 \cot\psi}{2} - A\right)\left(\sin\psi - \mu_1 \cos\psi\right) - cl \cdot \frac{y}{\sin\psi} = 0;$$

or, dividing by $l \sin\psi$,

$$n\gamma_1\left(\frac{y^2 \cot\psi}{2} - A\right)\left(1 - \mu_1 \cot\psi\right) - cy\left(1 + \cot^2\psi\right) = 0. \quad . \quad (13)$$

If aG makes an angle with the horizontal greater or less than ψ, we have, from (12), $n(P - \mu N)$ less than $cl \cdot \overline{aG}$, or the left side of equation (13) less than zero. The value of ψ must then make equation (13) a maximum.

If then we differentiate (13) with reference to $\cot\psi$ and put the first derivative equal to zero, we obtain

$$\frac{n\gamma_1 y^2}{2}\left(1 - \mu_1 \cot\psi\right) - n\gamma_1\mu_1\left(\frac{y^2 \cot\psi}{2} - A\right) - 2cy\cot\psi = 0. \quad (14)$$

Eliminating $\cot\psi$ from (13) and (14), we obtain

$$A = \frac{y}{2n\mu_1^2\gamma_1}\left[n\mu_1\gamma_1 y + 4c - 2\sqrt{2c(n\mu_1\gamma_1 y + 2c)(1 + \mu_1^2)}\right]. \quad (15)$$

Equation (15) gives the area A between the curve of the slope and any ordinate $da = y$. *It evidently holds good whether the area A is bounded by a curve or a broken line of any form.*

Values of ϕ_1, μ_1 and γ_1.—We give in the following Table the values of ϕ_1, μ_1, γ_1 for earth, sand and gravel.

Kind of Earth.	Angle of Repose, ϕ_1.	Coefficient of Friction, μ_1.	Mass of one cubic foot in pounds, γ_1.
Gravel, round....................	30°	0.58	100
" sharp....................	40	0.84	110
Sand, dry....................	35	0.70	100
" moist....................	40	0.84	110
" wet....................	30	0.58	125
Earth, dry....................	40	0.84	90
" moist....................	45	1.00	95
" wet....................	32	0.62	115

EXAMPLES.

(1) *A bank of loose earth without cohesion stands 30 ft. high with a slope of 50 ft. Find the coefficient of friction and the angle of repose.*

Ans. The horizontal projection of the slope is 40 ft. Hence $\mu_1 = \tan\phi_1 = \frac{30}{40} = 0.75$, and ϕ_1 is about 35°.

(2) *A bank of earth with vertical face is found to cave for a distance of 3 ft. below the surface. The same earth loose and without cohesion takes a slope of 1.25 to 1 horizontal. Find the slope after rupture. Also if the mass of a cubic foot is 100 lbs., find the coefficient of cohesion.*

Ans. From equation (6), page 257, since $\beta = 0$, the angle of rupture is $\psi = 45° + \frac{\phi_1}{2}$. The tangent of the angle of repose is $\mu_1 = \tan\phi_1 = 0.75$. Hence ϕ_1 is about 35° and ψ is about 62°.

From equation (8), page 257, since $h_0 = 3$ ft., $\gamma_1 = 100$ lbs. per cubic foot, $\phi_1 = 35°$,

$$c = \frac{100 \times 3(1 - \sin 35°)}{4 \cos 35°} = \frac{128}{3.28} = 39 \text{ lbs. per square foot.}$$

(3) *A bank of earth the same as in the preceding example has a height of 30 feet and a batter of 45°. Find the limiting height for the same slope and the factor of safety.*

Ans. From equation (9), page 257, since $h_0 = 3$ ft., $\beta = 45°$, $\phi_1 = 35°$, the limiting height is

$$h_1 = \frac{3(1 - \sin 35°) \cos 45°}{1 - \sin 80°} = \text{about } 60 \text{ ft.},$$

or the factor of safety is 2.

(4) *A bank of earth the same as in Example (2) is required to have a height of 30 ft. and a factor of safety of 2. Find the batter of the face.*

Ans. $\beta = 45°$.

(5) *A bank of earth with vertical face caves for a distance of 5 feet below the surface. The same earth loose and without cohesion takes a slope of 1.25 to 1 horizontal. The mass of a cubic foot is 100 lbs. Find the angle of rupture, the coefficient of cohesion. If the batter of the face is made 45° and the height 30 ft., find the factor of safety.*

Ans. The angle of repose is ϕ_1 = about 35°. The angle of rupture is ψ = about 62°. The coefficient of cohesion is $c = 65$ lbs. per square foot. From equation (10), page 258,

$$n = \frac{5(1 - \sin 35) \cos 45}{30(1 - \sin 80°)} = 3\tfrac{1}{3}.$$

(6) *Find the uniform batter-angle of the slope in the preceding example for a height of 30 ft. and a factor of safety of $3\tfrac{1}{3}$.*

Ans. From equation (11), page 258, we find $\beta = 45°$.

(7) *Find the natural curve of the slope in Example (5) for a factor of safety of 3 and a height of 40 feet.*

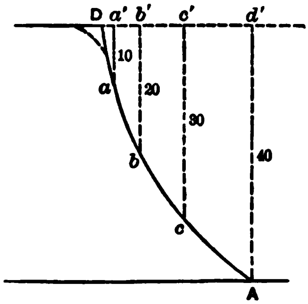
Ans. Since $\mu_1 = 0.75$, $c = 65$ lbs. per square foot, $n = 3$, $\gamma_1 = 100$, equation (15), page 259, becomes

$$A = \frac{y}{337.5}\left[225y + 260 - 2\sqrt{203\tfrac{1}{3}(225y + 130)}\right].$$

If we take $y =$ to 10, 20, 30, 40 ft., we have:

For $y = 10$, $A = 33$ sq ft.;
$y = 20$, $A = 167$ " "
$y = 30$, $A = 413$ " "
$y = 40$, $A = 777$ " "

We have then, considering the area between the slope and any ordinate as made up of trapezoids, as shown in the figure:



$$\frac{1}{2} \cdot 10 \,.\, Da' = 33, \quad \text{or} \quad Da' = 6.6 \text{ ft.};$$

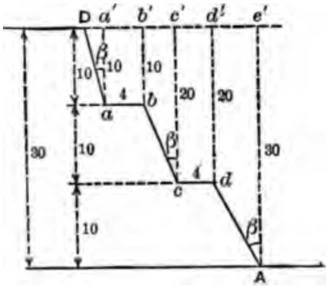
$$33 + \frac{10+20}{2} \,.\, a'b' = 167, \quad \text{or} \quad a'b' = 9 \quad \text{"}$$

$$167 + \frac{20+30}{2} \,.\, b'c' = 413, \quad \text{or} \quad b'c' = 9.8 \quad \text{"}$$

$$413 + \frac{30+40}{2} \,.\, c'd' = 777, \quad \text{or} \quad c'd' = 10.4 \quad \text{"}$$

We see from equation (15), page 259, that for small values of y A is negative, or, theoretically, the curve overhangs the slope. The equation should not be used for y less than h_0, and the upper part of the slope should be rounded off, as shown in the figure.

(8) *It is desired to cut a bank* 30 *feet high into three terraces as shown in the figure with a factor of safety of* 1.5. *The height of each terrace is to be* 10 *feet and there are to be two steps, ab and cd, each* 4 *feet wide. The mass per cubic foot is* $\gamma_1 = 100$ *lbs., and* ϕ_1 *and* h_0 *as found by experiment are* $\phi_1 = 31°$, $h_0 = 5$ *feet. Find the batter for each terrace.*



Ans. We have $\mu_1 = \tan \phi_1 = 0.6$, and from equation (8), page 257, $c = 71$, and equation (15), page 259, becomes

$$A = \frac{y}{108}\left(284 + 90y - 2\sqrt{189(90y + 142)}\right).$$

From this, when $y = 10$, $A = 27$; when $y = 20$, $A = 159$; and when $y = 30$, $A = 421$.

We have then

$$\frac{1}{2} \,.\, 10 \,.\, Da' = 27, \quad \text{or} \quad Da' = 5.4 \text{ ft.};$$

$$27 + 40 + \frac{10+20}{2} \,.\, b'c' = 159, \quad \text{or} \quad b'c' = 6.1 \quad \text{"}$$

$$159 + 40 + \frac{20+30}{2} \,.\, d'e' = 421, \quad \text{or} \quad d'e' = 8.9 \quad \text{"}$$

Hence we have for the batter-angles:

$$\text{For } Da, \ \tan\beta = \frac{5.4}{10}, \quad \text{or} \quad \beta = 28\tfrac{1}{2}°;$$

$$\text{For } bc, \ \tan\beta = \frac{6.1}{10}, \quad \text{or} \quad \beta = 31\tfrac{1}{2}°;$$

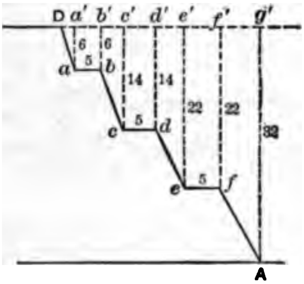
$$\text{For } dA, \ \tan\beta = \frac{8.9}{10}, \quad \text{or} \quad \beta = 41\tfrac{1}{2}°.$$

(9) *Design a terrace of four planes, the upper one being* 6 *feet in height, the lowest* 10 *ft., and the others* 8 *ft. The steps to be* 5 *feet*

in width, and the earth such that $h_0 = 3$ ft., $\gamma_1 = 100$, and $\mu_1 = 0.66$. Take the factor of safety at 2.

Ans. $c = 40$, $A = \frac{y}{178}\left[133y + 160 - 2\sqrt{115(133y + 80)}\right]$.

When $y = 6$, $A = 10.9$; $y = 14$, $A = 84.3$;
$y = 22$, $A = 236.4$; $y = 32$, $A = 569.2$.



$$\frac{1}{2}\cdot 6 \cdot Da' = 10.9, \text{ or } Da' = 3.63 \text{ ft.};$$

$$10.9 + 30 + \frac{6 + 14}{2}\cdot b'c' = 84.3,$$
$$\text{or } b'c' = 4.34 \text{ ft.};$$

$$84.3 + 70 + \frac{14 + 22}{2}\cdot d'e' = 236.4,$$
$$\text{or } d'e' = 4.5 \text{ ft.};$$

$$236.4 + 110 + \frac{22 + 32}{2}\cdot f'g' = 569.2,$$
$$\text{or } f'g' = 8.2 \text{ ft.}$$

We have then for the batter-angles:

$$\text{For } Da,\ \tan\beta = \frac{3.63}{6}, \text{ or } \beta = 31°;$$

$$\text{For } bc,\ \tan\beta = \frac{4.34}{8}, \text{ or } \beta = 28\tfrac{1}{2}°;$$

$$\text{For } de,\ \tan\beta = \frac{4.5}{8}, \text{ or } \beta = 29\tfrac{1}{2}°;$$

$$\text{For } fA,\ \tan\beta = \frac{8.2}{10}, \text{ or } \beta = 39\tfrac{1}{2}°.$$

(10) *Find the batter-angle β for a railway embankment 30 ft. high, 12 ft top base. Let $\gamma_1 = 100$ lbs. per cubic foot, $\phi_1 = 34°$, $h_0 = 4$ ft., and factor of safety 2. Let the locomotive weight be about 6000 pounds per linear foot of track.*

Ans. If the top base is 12 feet, the weight of locomotive causes a pressure of 6000 lbs. on 12 square feet, or 500 lbs. per square foot. This is equivalent to a mass of earth 5 feet high. We take then $h_1 = 35$ feet in equation (11), page 258, and have

$$\tan\frac{1}{2}\beta = \frac{1}{1.584}\left[0.829 + \sqrt{0.0286}\right] = 0.416.$$

Therefore $\frac{1}{2}\beta$ is about $22\tfrac{1}{2}°$, or $\beta = 45°$.

The embankment with this batter contains 47 cubic yards per linear foot, while with the natural slope of 34° it would contain 62 cubic yards per linear foot. There will then be a saving in cost of construction if the expense of protecting the slope to preserve the cohesion is not greater than the saving in embankment.

(11) *A railway cut is made in material for which $\gamma_1 = 100$ pounds per cubic foot, $\phi_1 = 34°$, $h_0 = 5$ ft. The depth of cut is $h_1 = 40$ ft. and the roadbed is 16 ft. Find the batter-angle for a factor of safety of 3.*

Ans. We have $\beta = 47°$. The cut with this batter contains 87 cubic yards per linear foot. If it had the natural slope, it would contain 111 cubic yards. There will then be a saving in cost if the expense of protecting the slope is less than the saving in excavation.

(12) *At Northfield, Vt., on the line of the Central Vermont R. R. is a retaining wall 15 ft. high, top base 2 ft., bottom base 6 ft. The wall is composed of large blocks of limestone without cement, the density of the masonry about 170 lbs. per cubic foot. The earth surface is horizontal and level with the top of the wall; angle of repose 38°, and density of earth 90 lbs. per cubic foot. The front face of the wall has a batter of 1 inch horizontal for every foot of height. This wall is over 30 years old and in as good condition as when laid. Investigate its stability and check results of computation by graphic construction.*

Ans. We have $h = h_1 = 15$ ft., $\alpha = 0°$, $\delta = 170$ lbs. per cubic foot, $\gamma_1 = 90$ lbs. per cubic foot, $\phi_1 = 38°$, $b_1 = 2$ ft., $b_2 = 6$ ft., $\tan\beta = \frac{2.75}{15}$ or $\beta = 10°\ 23'$.

Take a section of the wall one foot in length, so that $l = 1$. Then from page 254, Case 1, we have

$$P = \frac{90 \times 15^2}{2}\sqrt{\frac{1}{0.967} - \frac{2\ 463}{2.6}} = 2983 \text{ lbs.}$$

We have also from equation (8), page 254,

$$\tan\epsilon = \tan^2 26°\ \cot 10°\ 23', \quad \text{or} \quad \epsilon = 52°\ 26'.$$

Then from equation (9), page 254, $\theta = 27°\ 11'$. The angle of P with the horizontal is then $(\theta + \beta) = 37°\ 34'$, and the horizontal and vertical components of P are

$$H = P\cos(\theta + \beta) = 2364 \text{ lbs.};$$

$$V = P\sin(\theta + \beta) = 1814 \text{ lbs.}$$

The weight of a section one foot in length is

$$W = 11200 \text{ lbs.}$$

If we take the coefficient of static sliding friction $\mu = 0.66$ (page 229), we have from equation (I), page 233, for the factor of safety for sliding

$$n = \frac{0.66(11200 + 1814)}{2364} = 3.6,$$

or, if we neglect V, $n = 3.1$. There is therefore ample security against sliding. If there are no through joints, there is in any case no possibility of sliding.

From equation (5), page 233, we have $s_2 = 3.3$ ft., and from equation (II), page 233, $e = 2.1$ ft. The resultant of P and W, therefore, cuts the base within the middle third and just within the middle third. *The proportions are then nearly economic.* Thus from equation (III), page 234, we have $b_2 = 5.86$ ft., while the bottom base as built is 6 ft.

From equation (7), page 234, we have for the greatest unit compression two tons per square foot, which, as we see from page 229, is abundantly safe.

(13) *In the preceding example, let the back be vertical. Find the bottom base. Check the computation by graphic construction.*

Ans. In this case, $\beta = 0$. From page 254, Case 2, we find the earth pressure horizontal or $\theta = 0$, $\beta = 0$, and if we take a section of wall one foot in length, so that $l = 1$,

$$P = \frac{90 \times 15^2}{2}\tan^2\left(45° - \frac{38°}{2}\right) = 2410 \text{ lbs.}$$

From equation (III), page 234, we have for the bottom base when $e = \frac{1}{3}b_2$, or for economic proportions,

$$b_2 = 4.8 \text{ ft.}$$

From (I), page 233, we have the factor of safety for sliding, $n = 2.4$.

From equation (6) we have for the greatest unit compression 1.8 tons per square foot, which is much less than the allowable safe stress (page 229).

(14) *Find the bottom base of a trapezoidal wall of granite ashlar with vertical back, 20 feet high, to retain an embankment, the earth surface being horizontal and level with the top of the wall; $\phi_1 = 33° 40'$, $\gamma_1 = 100$ lbs. per cubic foot. Check the computation by graphic construction.*

Ans. In this case, $\beta = 0$. From page 254, Case 2, we find the earth pressure horizontal, and taking a section of wall one foot in length, or $l = 1$,

$$P = \frac{100 \times 20^2}{2} \tan^2 28° 15' = 5774 \text{ lbs.}$$

From equation (III), page 234, we have for the bottom base for economic proportions, or for $e = \frac{1}{3}b_2$,

$$b_2 = -\frac{1}{2}b_1 + \sqrt{\frac{5b_1^2}{4} + \frac{6P \cdot \frac{1}{3}h}{\delta h}}.$$

If we take the top base $b_1 = 2$ ft. and $\delta = 165$ lbs. per cubic foot (page 229), we have $b_2 = 7.66$ ft.

From equation (6), page 234, the greatest unit compression is about 2 tons per square foot, which is much less than the allowable safe stress (page 234).

(15) *Same as Example (14), with back batter $\beta = 8°$. Check the computation by graphic construction.*

Ans. $P = 6420$ lbs., $\theta = 18° 9'$, $H = 5758$ lbs., $V = 2825$ lbs., $b_2 = 7.9$ ft. Greatest unit compression 2.4 tons per square foot, which is much less than the allowable safe stress (page 229).

(16) *A rubble wall of limestone, 15 ft. high, retains an earth-filling which supports a double-track railway. The top base is $b_1 = 3.5$ ft. Find the bottom base when $\gamma_1 = 100$, $\phi_1 = 33° 40'$, $\beta = 8°$, $\delta = 170$ lbs. per cubic foot.*

Ans. If we take the train load at 6000 lbs. per linear foot, and top base of the fill 15 ft., the pressure per square foot on the top is 400 lbs., which is equivalent to a column of earth 4 ft. high. We have then $h = 15$ ft., $h_1 = 15 + 4 = 19$ ft., and

$$P = 5795 \text{ lbs.}, \quad \theta = 18°, \quad H = 5200 \text{ lbs.}, \quad V = 2540 \text{ lbs.},$$

$b_2 = 7$ ft. Greatest unit compression 2.3 tons per square foot, which is much less than the allowable safe stress (page 229).

(17) *Find the bottom base for a retaining wall 20 ft. high, back batter $\beta = 8°$, $\delta = 170$ lbs. per cubic foot. Earth surface inclined to horizontal at angle of repose $b_1 = 33° 40'$, $h_1 = 20$ ft., $\gamma_1 = 100$ lbs. per cubic foot.*

Ans. In this case we have, from page 255, Case 4, $\epsilon = 21° 22'$, $\theta = 32° 28'$, $P = 21740$ lbs., $H = 16522$ lbs., $V = 13230$ lbs.

If we take the top base $b_1 = 2$ ft., we have, from equation (III), page 234, $b_2 = 9.6$ ft. The greatest unit stress of compression is 1.7 tons per square foot.

(18) *The San Mateo dam, California, is built of concrete weighing about* 150 *pounds per cubic foot. The height is* $h = 170$ *ft., top base* $b_1 = 20$ *ft., bottom base* $b_2 = 176$ *ft., back batter* 1 *to* 4 *or* $\tan \beta = 0.25$. *Investigate the stability for depth of water* $h_1 = 165$ *ft.*

Ans. We have for a section one foot in length

$$V = 212700 \text{ lbs.}, \quad H = 850780 \text{ lbs.}, \quad W = 2499000 \text{ lbs.}$$

There are no through joints in this dam, and therefore no investigation for sliding is needed. If, however, we take the coefficient of static sliding friction $\mu = 0.66$ (page 229), we have from equation (I), page 236, $n = 2$.

If the dam is empty, we have from equation (5), page 237, $s_2 = 75$ ft. The weight then cuts the base near the middle and well within the middle third.

From equation (II), page 237, we have, even when we take ice-thrust into account, $e = 86$ ft. The resultant of the weight, pressure and ice-thrust then cuts the base within the middle third.

Hence from equation (7), page 238, we have for dam empty the greatest unit stress of compression 11 tons per square foot, and for dam full and ice-thrust 8 tons per square foot.

The dam as built is then stable and safe even for a cold climate, and even for through joints.

(19) *Design a dam of sandstone ashlar,* 60 *ft. high, top base* 9 *ft., depth of water* 57 *feet.* ✓

Ans. We have $h = 60$ ft., $h_1 = 57$ ft., $b_1 = 9$ ft., $\gamma = 62.5$ lbs. per cubic foot, and, from page 239, $\delta = 150$ lbs. per cubic foot, $C = 20$ tons per square foot, $\mu = 0.6$.

From page 229 we take the back vertical for economic section. Hence $\beta = 0$.

From equation (III), page 237, we have for economic proportions for the bottom base $b_2 = 32.7$ ft. and hence $A = 1250$ square feet.

Then from equation (6), page 237, the greatest compressive stress for reservoir full is $p = 5.7$ tons per square foot. For reservoir empty s_2 is always greater than $\frac{1}{3}b_2$ when back is vertical (page 238), and the unit stress is still less.

We have then for a foot of length of the dam, $W = 187500$ lbs., $H = 101530$ lbs., and from equation (I), page 236, if there is no ice-thrust, we have for the factor of safety for sliding $n = 1.1$. This is small, but if there are no through joints the dam cannot slide.

But now, if we suppose the ice-thrust of $T = 40000$ lbs. per foot to act, we must test and see if the dam with bottom base $b_2 = 32.7$ ft. is still safe.

From (5), page 237, we have $s_2 = 11.6$ ft., and from (II), page 237, using this value of s_2, we obtain $e = -1.35$ ft. The minus sign shows that the resultant passes outside of the base. The dam would therefore rotate under the ice-thrust. We must find b_2 therefore from (III′), page 238. This gives us $b_2 = 36$ ft. and $A = 1350$ sq. ft., $W = 202500$ lbs.

We have now for the factor of safety for sliding $n = 0.9$. This is less than unity, and hence when the ice-thrust acts, the wall must depend for its safety entirely upon the fact that there are no through joints. It would be better, then, to give the dam a back batter of, say, $\tan \beta = 0.25$.

If we do this, we have from (III), page 237, $b_2 = 43.6$ ft. and $A = 1578$ sq. ft. From (5) and (II), page 237, we then obtain $s_2 = 21$ ft. and $e = 8$ ft. Then from (8), page 238, we have $p = 10.9$ tons per square foot, so that so far as rotation and compression are concerned the dam is safe even with ice-thrust acting.

We have now from (I), page 236, for the factor of safety for sliding, when the ice-thrust acts, $n = 1.1$. We should then have no through joints in the dam.

(20) *The height of the proposed Quaker Dam, New York, is* 170 *feet, top thickness* 20 *feet, specific mass of the masonry* 2.5, *depth of*

water 163 *feet. Find the economic section for allowable compression of* 10 *tons per square foot.*

Ans. We have $b_1 = 20$ ft., $h_1 = 163$ ft., $h = 170$ ft., $\frac{\delta}{\gamma} = 2.5$, $\gamma = 62.5$ lbs. per cubic foot, $\delta = 156.25$ lbs. per cubic foot, $T = 40000$ lbs. per foot, $C = 20000$ lbs. per square foot, $\mu = 0.6$.

1st. Ice-thrust Neglected.—Let us first neglect the ice-thrust.

From equation (I), page 240, we have for the height h_2 of the first rectangular sub-section if the water is level with the top, $h_2 = 32$ ft. As the water is not level with the top, h must be greater than this. In equation (I), page 240, $\delta b_1{}^2 h_2 = \gamma(h_2 - a)^3$, if we put $a = 7$ ft., $h_2 = h_1 + 7$, and insert the values of δ, b_1 and γ, we have

$$625000\, h_2 + 4375000 = 62.5 h_1{}^3.$$

Solving this equation, we have for $e = \frac{1}{3} b_1 = 6.66$ ft., $h_1 = 34.7$ ft. Hence $h_2 = 41.7$ ft. and $A_1 = 834$ sq. ft. When the dam is empty $s = \frac{b_1}{2} = 10$ ft. We have then from (7), page 238, when the dam is empty, the unit compression $p = 3.26$ tons per square foot on back edge, and from (6), page 237, when the dam is full, $p = 6.52$ tons per square foot on front edge.

Below $h_1 = 34.7$ ft. we have the back vertical and the face battered and the second sub-section begins.

Let us take for the height of the next trial section $h_2 = 15.3$ ft. Then $h_1 = 34.7 + 15.3 = 50$ ft., $A_1 = 834$ sq. ft., $b_1 = 20$ ft., $s_1 = 10$ ft., $\beta = 0$, $\frac{\delta}{\gamma} = 2.5$. From (III), page 243, when $e = \frac{1}{3} b_2$, we have $b_2 = 26.2$ ft., and hence $e = 8.7$ ft. The area of this trial section is then $A_2 = 353$ sq. ft. We have now from (2) and (II), page 242, $s_2 = 11.6$ ft. and $s = 10.5$ ft. Then from (7) and (6), page 238, the unit compression $p = 5.66$ tons per square foot on back edge for dam empty and $p = 7.08$ tons per square foot on front edge for dam full.

Take for the height of the next trial section $h_2 = 20$ ft. Then $h_1 = 70$ ft., $A_1 = 11.87$ sq. ft., $b_1 = 26.2$ ft., $s_1 = 10.5$ ft., $\beta = 0$, $\frac{\delta}{\gamma} = 2.5$.

Just as before, from (III), page 243, when $e = \frac{1}{3} b_2$, we now have $b_2 = 37.4$ ft., and hence $e = 12.5$ ft., and $A_2 = 636$ sq. ft. Then from (2) and (II), page 242, $s = 12.4$ ft. Then from (6), page 238, the unit compression is $p = 7.62$ tons per square foot on back edge for dam empty and $p = 7.62$ tons per square foot on front edge for dam full. Since for $h_1 = 70$ we have $s = 12.4 = \frac{1}{3} b_2$, this is the limit of the second sub-section.

Below $h_1 = 70$ ft. we must batter both front and back. If then we take $h_2 = 20$ ft. for the next trial section, we have $h_1 = 90$ ft., $A_1 = 18.23$ sq. ft. $b_1 = 37.4$ ft., $s_1 = 12.4$ ft., $\frac{\delta}{\gamma} = 2.5$.

From (IV), page 244, we have then, when $e = \frac{1}{3} b_2 = s$, $b_2 = 53.4$ Hence $s = e = 17.8$ and $A_2 = 908$ sq. ft., and from (V), page 244, we obtain tan $\beta = 0.114$. Then from (4), page 244, the compression on front edge for dam full or on back edge for dam empty is $p = 7.99$ tons per square foot.

Take $h_2 = 20$ ft. for the next trial section. Then $h_1 = 110$ ft., $A_1 = 2731$ sq. ft., $b_1 = 53.4$ ft., $s_1 = 17.8$ ft., $\frac{\delta}{\gamma} = 2.5$, and we have from (IV), page 244 (*a*), $b_2 = 67.5$ ft , hence $A_2 = 1218$ sq. ft., $e = s = 22.5$ ft., and from (V), page 244, tan $\beta = 0.05$. From (4), page 243 (*a*), the compression on front and back edge for dam full and empty is $p = 9.14$ tons per square foot.

Take $h_2 = 20$ ft. for the next trial section. Then $h_1 = 130$ ft., $A_1 = 3949$ sq. ft., $b_1 = 67.5$ ft., $s_1 = 22.5$ ft., $\frac{\delta}{\gamma} = 2.5$. We find then for this section $b_2 = 81.6$ ft., $A_2 = 1490$ sq. ft., $e = s = 27.2$ ft., $\tan\beta = 0.036$, $p = 10.4$ tons per square foot.

Below $h_1 = 130$ ft., then, the fourth sub-section begins and we must use equation (VI), page 245.

Take $h_2 = 20$ ft. for the next trial section. Then $h_1 = 150$ ft., $A_1 = 5439$ sq. ft., $b_1 = 81.6$ ft., $s_1 = 27.2$ ft., $\gamma = 62.5$ lbs. per cubic foot, $\delta = 156.25$ lbs. per cubic foot. Then, from (VI), page 245, $b_2 = 106.7$ ft. and hence $A_2 = 1883$ sq. ft., and from (5), page 244 (*a*), $e = s = 38$ ft. From (V) we have $\tan\beta = 0.18$.

For the remaining depth $h_2 = 13$ ft., $h_1 = 163$ ft., $A_1 = 7322$ sq. ft., $b_1 = 106.7$ ft., $s_1 = 38$ ft., and we find $b_2 = 123.6$ ft., $A_2 = 1497$ sq. ft., $e = s = 45.4$ ft., $\tan\beta = 0.00$.

We have then the following Table:

h	h_1	b	A	$\tan\beta$	e	s	p back	p front
41.7	34.7	20	834	0	6.6	10.0	3.26	6.52
57	50	26.2	1187	0	8.7	10.5	5.66	7.08
77	70	37.4	1823	0	12.5	12.4	7.62	7.62
97	90	53.4	2731	0.114	17.8	17.8	7.99	7.99
117	110	67.5	3949	0.05	22.5	22.5	9.14	9.14
137	130	81.6	5439	0.036	27.2	27.2	10.4	10.4
157	150	106.7	7322	0.18	38.0	38.0	10.0	10.0
170	163	123.6	8819	0.00	45.4	45.4	10.0	10.0

In this Table the first column contains the height h of the dam in feet above the base of each sub-trapezoid, the second the depth of water h_1 in feet above the base of each sub-trapezoid, the third the base b in feet of each sub-trapezoid, the fourth the total area A in square feet above that base, the fifth the tangent of the back batter-angle $\tan\beta$, the sixth and seventh the distances e and s in feet from front and back edges to where the resultant cuts the base of each sub-trapezoid for dam full and empty, the eighth and ninth the unit stress p of compression at those edges in tons per square foot.

Comparing with Ex. (18), we see that the San Mateo dam, 170 ft. high, has about 88 per cent more material than this economic section of the same height.

2d. Ice-thrust taken into Account.—Let us now consider the same dam, taking the ice-thrust into account.

From equation (I′), page 241, putting $h_2 = h_1 + 7$ and $a = 7$, we have, after substituting $\gamma = 62.5$, $\delta = 156.25$, $b_1 = 20$, $C = 20000$, $T = 40000$,

$$\frac{h_1^3}{300} + \frac{6250 h_1^2}{60000} = 4.258 h_1 = 64.896, \quad \text{or} \quad h_1 = 11 \text{ ft.}$$

Hence $h_2 = 18$ ft. and $A_1 = 360$ sq. ft., $e = 1.9$ ft., $p = 10$ tons per square foot.

Below $h_1 = 11$ ft. we have the back vertical and face battered, and the second sub-section begins.

Let us take for the height of the next trial section $h_2 = 23.7$ ft. Then $h_1 = 34.7$ ft., $A_1 = 360$ sq. ft., $b_1 = 20$ ft., $s_1 = 10$ ft., $\beta = 0$, $\gamma = 62.5$, $\delta = 156.25$, $C = 20000$, $T = 40000$. From (III′) we have $b_2 = 28.76$ ft.; hence $A_2 = 577.8$ sq. ft., and from (3), page 242, $e = 4.8$ ft. From (2) and (II), page 242, we then have $s = 11.4$ ft.

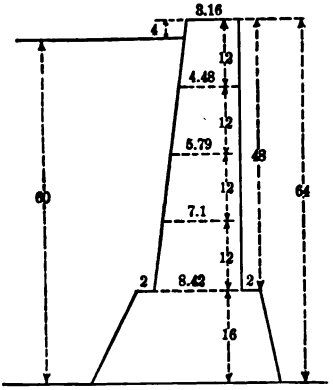
Take $h_2 = 15.3$ ft. for the height of the next trial section. Then $h_1 = 50$ ft., $A_1 = 938$ sq. ft., $b_1 = 28.76$ ft., $s_1 = 11.4$ ft., $\beta = 0$, and we can find b_2, A_2 and s_2 for this section.

We can then take $h_2 = 20$ ft., and so on, until we arrive at a section for which $s = \frac{1}{3} b_2$.

Below this section we must batter face and back, still using (III'), page 243, for b_2 and finding $\tan\beta$ from (V), page 244.

The student should complete the example.

(21) *The Bear Valley dam in the San Bernardino Mountains, California, is an arch dam about 450 ft. long, constructed of granite ashlar, height $h = 64$ ft., radius $r = 300$ ft., top base $b_1 = 3.17$ ft., bottom base $b_2 = 20$ ft., depth of water $h_1 = 60$ ft., face vertical. Other dimensions as shown in the figure. Examine its stability.*



Ans. We have from the given dimensions and from equation (2), page 246 (*d*), neglecting the ice-thrust *T*,

for distance from top	= 12	24	36	48	64	ft.
$h_1 =$	8	20	32	44	60	"
$t =$	4.48	5.79	7.1	8.42	20	"
$C =$	16.74	32.38	42.25	43.05	28.12	

tons per square foot.

From page 229, the allowable unit compression C ought not to exceed 30 tons per square foot. The dam as built has then a higher unit stress than good practice would consider allowable.

(22) *Design an arch dam of the same height and radius as the Bear Valley dam, Ex. (21), and same depth of water, for an allowable compressive stress of 25 tons per square foot.*

Ans. We have $h = 64$ ft., $h_1 = 60$ ft., $r = 300$ ft., $C = 50000$ lbs. per square foot, $\gamma = 62.5$ lbs. per cubic foot.

In default of local or practical considerations to guide us in choice of the top base b_1, let us suppose an ice-thrust of $T = 40000$ lbs. per foot.

Then from (3), page 246, we have for the top base

$$b_1 = \frac{300 \times 40000}{50000 \times 64} = 3.75 \text{ ft.}$$

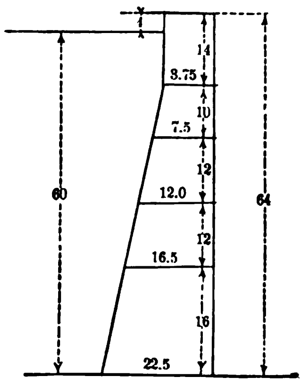
1st. Without Ice-thrust.—Let us take then $b_1 = 3.75$ ft., and suppose first that there is no ice-thrust.

Then from (4), page 246, neglecting *T*, we have for the distance h_2 below the water level for which the cross-section may be made rectangular,

$$h_2 = \frac{50000 \times 3.75}{62.5 \times 300} = 10 \text{ ft.}$$

The dam then is rectangular for 14 ft. below the top. Below this point we must increase the thickness as the depth of water increases. We have then from (2), page 246, neglecting *T*,

for distance from top	= 14	24	36	48	64	ft.
$h_1 =$	10	20	32	44	60	"
$t =$	3.75	7.5	12	16.5	22.5	"



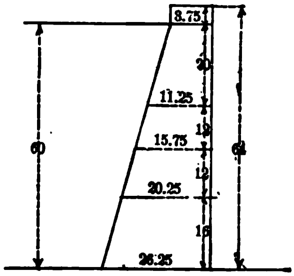
If we make the face vertical and batter the back, we have then a cross-section as shown

in the figure 3.75 ft. thick for the first 14 feet, and then with a back batter of $\frac{18.75}{50}$, or $\tan \beta = 0.375$.

2d. With Ice-thrust.—If we consider the ice-thrust T as acting, then we have b_1 at least 3.75 ft. as already found.

From (4), page 246, taking $T = 40000$, we have for the distance h_1 below the water level for which the cross-section may be made rectangular

$$h_1 = \frac{50000 \times 3.75}{62.5 \times 300} - \frac{40000}{62.5 \times 64} = 0.$$



The dam then is rectangular for 4 feet below the top. Below this point we must increase the thickness as the depth of water increases. We have then from (2), page 246 (*d*), for

dist. from top	$=4$	24	36	48	64	ft.
h_1	$=0$	20	32	44	60	"
t	$=3.75$	11.25	15.75	20.25	26.25	"

If we make the face vertical and batter the back, we have then a cross-section as shown in the figure 3.75 ft. thick for the first 4 feet, and then with a back batter of $\frac{22.5}{60}$, or $\tan \beta = 0.375$.

CHAPTER II.

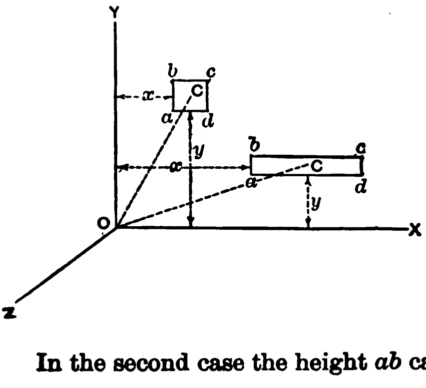
APPLICATIONS OF STATICS—STRENGTH AND ELASTICITY OF MATERIALS.

MOMENT OF INERTIA OF AN AREA. RADIUS OF GYRATION. DETERMINATION OF MOMENT OF INERTIA OF AREAS. STRESS AND STRAIN. EXPERIMENTAL LAWS. COEFFICIENT OF ELASTICITY. WORK AND COEFFICIENT OF RESILIENCE. EQUILIBRIUM OF A DEFLECTED BEAM. SHEARING FORCE AND SHEARING STRESS. BENDING MOMENT. NEUTRAL AXIS. RESISTING MOMENT. COEFFICIENT OF RUPTURE FOR FLEXURE. TABLE OF PROPERTIES OF MATERIALS. FACTOR OF SAFETY AND WORKING STRESS. VARIABLE WORKING STRESS. STRENGTH OF PIPES AND CYLINDERS. RIVETED JOINTS. THEORY AND PRACTICE OF RIVETING. DESIGNING OF BEAMS. BREAKING WEIGHT. SHAPE FOR UNIFORM STRENGTH. THEORY OF PINS AND EYEBARS. TORSION. COMBINED STRESSES. STRESS DUE TO TEMPERATURE.

Moment of Inertia of an Area.—The term "**moment of inertia of an area**" is used to designate a quantity which occurs so frequently in the application of statics to the strength and elasticity of materials that a special name and symbol for it is essential. Before taking up such application, then, it is necessary to define what is meant by the term and to show how the quantity it stands for may be computed. The use made of it will appear later.

Definition of Moment of Inertia of an Area.—Any indefinitely small area we call an **elementary area**. Thus the rectangular areas *abcd* are elementary areas if in the one case the height and breadth *ab* and *cb* are indefinitely small, and if in the other case, whatever the breadth *bc*, the height *ab* is indefinitely small. *An elementary area, then, has one or both of its dimensions indefinitely small.*

Take O as origin and draw the co-ordinate axes OX and OY in the plane of the areas, parallel to the base and height. Then in the first case, since both dimensions are indefinitely small, *they can be neglected with reference to any finite distance*. The perpendicular x from *ab* on OY is then the distance of the area *abcd* from the axis of Y, or the same as the distance of the centre of mass C of the area from the axis of Y, and the perpendicular y from *ad* on OX is the same as the distance of the centre of mass C of the area from the axis of X.



In the second case the height *ab* can be neglected with reference

to any finite distance, and the perpendicular y from ad on OX is the same as the distance of the centre of mass C of the area from the axis of X. The perpendicular x from C on OY is the distance from the axis of Y.

In either case, *the product of the elementary area by the square of its distance from any axis in the plane of the area is called the moment of inertia of the elementary area with reference to that axis.*

Thus if a is the elementary area, ax^2 is its moment of inertia with reference to OY in its plane, and ay^2 is its moment of inertia with reference to OX in its plane.

In the same way if r is the distance OC of the elementary area from the axis of Z, ar^2 is its moment of inertia with reference to the axis OZ *perpendicular to the plane of the area.* This is called the **polar moment of inertia of the area with reference to** OZ. But evidently $ar^2 = ax^2 + ay^2$. Hence, *the polar moment of inertia is equal to the sum of the moments of inertia with reference to any two co-ordinate axes in the plane of the area.*

Now any area may be considered as made up of an indefinitely great number of elementary areas. *The moment of inertia of an area with reference to any axis is then the sum of the moments of inertia of all its elementary areas.*

Thus the moment of inertia of any area with reference to the axes of X and Y in the plane of the area is given by

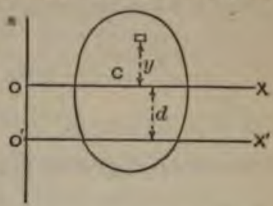
$$\Sigma ay^2 \quad \text{and} \quad \Sigma ax^2,$$

and the polar moment of inertia, or the moment of inertia with reference to the axis of Z at right angles to the plane of the area, is given by

$$\Sigma ar^2 = \Sigma a(x^2 + y^2) = \Sigma ax^2 + \Sigma ay^2,$$

or the sum of the moments of inertia with reference to the two co-ordinate axes in the plane of the area.

If the axis is taken through the centre of mass C of the area, we denote the corresponding moment of inertia by I. If it is not taken through the centre of mass, we call it an **eccentric axis**, and we denote the corresponding moment of inertia by I'.



Let OX be an axis which passes through the centre of mass C of a given area in its plane, and $O'X'$ a parallel eccentric axis, at a distance d from the first axis, also in the plane of the area.

Then the moment of inertia of the area with reference to OX is

$$I = \Sigma ay^2,$$

and the moment of inertia of the area with reference to $O'X'$ is

$$I' = \Sigma a(y + d)^2 = \Sigma ay^2 + 2d\Sigma ay + d^2\Sigma a.$$

But since OX passes through the centre of mass of the area, $\Sigma my = 0$ (page 17), where m is the mass of an elementary area. But $m = \delta a$, where a is the area and δ the surface density. Hence $\Sigma\delta ay = \delta\Sigma ay = 0$, or $\Sigma ay = 0$. Therefore, since $\Sigma a = A =$ the entire area, we have

$$I' = \Sigma ay^2 + Ad^2 = I + Ad^2.$$

That is, *the moment of inertia of an area with reference to an eccentric axis is equal to the moment of inertia with reference to a parallel axis through the centre of mass plus the area into the square of the distance between the two axes.*

Radius of Gyration of an Area.—The square root of the quotient obtained by dividing the moment of inertia of an area with reference to any axis by the area is called the **radius of gyration** of the area with reference to that axis. We denote the radius of gyration by κ. Then by definition

$$\kappa' = \sqrt{\frac{I'}{A}} \quad \text{and} \quad \kappa = \sqrt{\frac{I}{A}},$$

where κ' and I' indicate an eccentric axis, and κ and I an axis through the centre of mass.

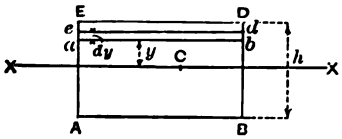
We have then

$$A\kappa^2 = I, \quad \text{or} \quad A\kappa'^2 = I'.$$

That is, *the radius of gyration of an area is that distance at which, if we suppose the entire area to be concentrated into a point, the moment of inertia is the same as for the given area.*

Determination of Moment of Inertia of an Area.—To determine the moment of inertia of an area with reference to any axis, we have simply to perform the summation indicated by Σax^2, or Σay^2, or Σar^2.

(1) **Moment of Inertia of the Area of a Rectangle.**—Let $ABDE$ be a rectangle of base $AB = b$ and height $BD = h$. Take the axis CX through the centre of mass C in the plane of the rectangle and parallel to the base b. Let $abde$ be an elementary area or strip parallel to the base at a distance y from the axis. Then the height of this strip is dy and its area is $a = bdy$ and its moment of inertia is $ay^2 = by^2dy$. The moment of inertia of the rectangle with reference to the axis CX is then, since the area of the rectangle is $A = bh$,



$$I_x = \int_{-\frac{h}{2}}^{+\frac{h}{2}} by^2dy = \frac{bh^3}{12} = A \cdot \frac{h^2}{12}.$$

The radius of gyration is $\kappa_x = \sqrt{\frac{I}{A}} = \frac{h}{2\sqrt{3}}.$

If we take the axis in the plane of the rectangle through the centre of mass C and parallel to the height h, we have in the same way

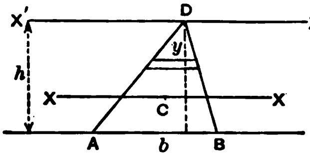
$$I_y = \int_{-\frac{b}{2}}^{+\frac{b}{2}} hx^2dx = \frac{hb^3}{12} = A \cdot \frac{b^2}{12}, \qquad \kappa_y = \sqrt{\frac{I}{A}} = \frac{b}{2\sqrt{3}}.$$

For the polar axis through the centre of mass C at right angles to the plane we have

$$I_z = A \cdot \frac{h^2 + b^2}{12} = A \cdot \frac{d^2}{12}, \quad \kappa_z = \sqrt{\frac{I}{A}} = \frac{d}{2\sqrt{3}},$$

where $d = \sqrt{h^2 + b^2}$ is the diagonal of the rectangle.

(2) **Moment of Inertia of the Area of a Triangle.**—Let ABD be a triangle of base $AB = b$ and height h. Take the axis $X'X'$ through the apex parallel to the base and in the plane of the area.



Take an elementary strip at a distance y from XX' parallel to the base. We have for the length x of this strip

$$x : y :: b : h, \quad \text{or} \quad x = \frac{by}{h}.$$

The area of the strip is then $a = xdy = \frac{bydy}{h}$, and the moment of inertia of the triangle with reference to XX' is then, since the area of the triangle is $A = \frac{bh}{2}$,

$$I_x' = \int_0^h \frac{b}{h} y^2 dy = \frac{bh^3}{4} = A \cdot \frac{h^2}{2}, \quad \text{and} \quad \kappa_x' = \sqrt{\frac{I'}{A}} = \frac{h}{\sqrt{2}}.$$

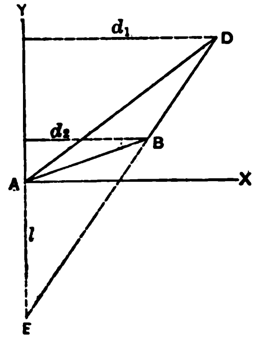
We have then for the moment of inertia with reference to the axis XX through the centre of mass C, parallel to the base and in the plane of the area,

$$I_x = I_x' - A\left(\frac{2}{3}h\right)^2 = A\frac{h^2}{18}, \quad \text{and} \quad \kappa_x = \sqrt{\frac{I}{A}} = \frac{h}{3\sqrt{2}}.$$

Again, we have for the moment of inertia with reference to the axis coinciding with the base AB,

$$I_b' = I + A\left(\frac{1}{3}h\right)^2 = A\frac{h^2}{6}, \quad \text{and} \quad \kappa_b = \sqrt{\frac{I'}{A}} = \frac{h}{\sqrt{6}}.$$

Take the axis AY through the vertex A in the plane of the triangular area ABD. Drop the perpendiculars d_1 and d_2 from D and B upon AY. Produce the side DB to intersection E with AY, and let the distance $AE = l$.



Let A_1 be the area of the triangle AED so that $A_1 = \frac{ld_1}{2}$. The moment of inertia of this triangle with reference to the axis AY conciding with the base AE is, as we have just seen, $I_1' = A_1\frac{d_1^2}{6} = \frac{ld_1^3}{12}$.

Let A_2 be the area of the triangle AEB, so that $A_2 = \frac{ld_2}{2}$. The moment of inertia of this triangle with reference to the axis AY coinciding with the base AE is $I_2' = A_2\frac{d_2^2}{6} = \frac{ld_2^3}{12}$.

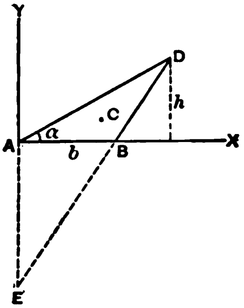
Hence the moment of inertia of the triangle ABD with reference to the axis AY is

$$I_y' = I_1' - I_2' = \frac{l}{12}(d_1^3 - d_2^3) = \frac{l}{2}(d_1 - d_2) \cdot \frac{1}{6}(d_1^2 + d_1 d_2 + d_2^2).$$

But $\frac{l}{2}(d_1 - d_2)$ is the area A of the triangle ABD. Hence we have

$$I_y' = \frac{A}{6}(d_1^2 + d_1 d_2 + d_2^2).$$

If the axis AY is at right angles to the side $AB = b$, and α is the angle DAB at A, then we have $d_1 = \frac{h}{\tan\alpha}$, $d_2 = b$, and



$$I_y' = \frac{A}{6}\left(b^2 + \frac{bh}{\tan\alpha} + \frac{h^2}{\tan^2\alpha}\right).$$

The distance from A to the centre of mass C is

$$\frac{b}{2} + \frac{1}{3}\left(\frac{h}{\tan\alpha} - \frac{b}{2}\right) = \frac{1}{3}\left(b + \frac{h}{\tan\alpha}\right).$$

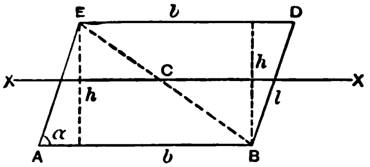
The moment of inertia with reference to an axis in the plane through the centre of mass C parallel to AY is then

$$I_y = I' - A \cdot \frac{1}{9}\left(b + \frac{h}{\tan\alpha}\right)^2 = \frac{A}{18}\left(b^2 - \frac{bh}{\tan\alpha} + \frac{h^2}{\tan^2\alpha}\right).$$

For the polar axis through the centre of mass C at right angles to the plane we have then

$$I_z = \frac{A}{18}\left(h^2 + b^2 - \frac{bh}{\tan\alpha} + \frac{h^2}{\tan^2\alpha}\right).$$

(3) **Moment of Inertia of the Area of a Parallelogram.**—We can divide the parallelogram $ABDE$ into two triangles by the diagonal EB.



The moment of inertia of the triangle ABE with reference to the axis ED is, as we have already found, $I_b' = \frac{bh^3}{4}$. The moment of inertia of the triangle EDB with reference to the axis ED is, as already found, $I_b' = \frac{bh^3}{12}$. The moment of inertia of the parallelogram with reference to the axis ED or AB is then

$$I_b' = \frac{bh^3}{3} = A \cdot \frac{h^2}{3}, \quad \text{and} \quad \kappa_b' = \sqrt{\frac{I'}{A}} = \frac{h}{\sqrt{3}}.$$

The moment of inertia of the parallelogram with reference to the axis XX in the plane through the centre of mass C parallel to the base AB is then

$$I_x = I' - A\left(\frac{h}{2}\right)^2 = A\frac{h^2}{12}, \quad \text{and} \quad \kappa_x = \sqrt{\frac{I}{A}} = \frac{h}{2\sqrt{3}},$$

or the same as for a rectangle.

In the same way if α is the acute angle at A, we have for the moment of inertia with reference to the axis AE or BD,

$$I_l' = A \cdot \frac{b^2 \sin^2\alpha}{3}, \quad \text{and} \quad \kappa_l' = \sqrt{\frac{I'}{A}} = \frac{b\sin\alpha}{\sqrt{3}},$$

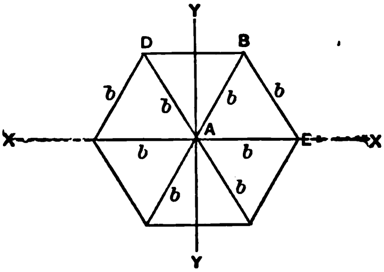
and with reference to the axis parallel to AE in the plane through the centre of mass C,

$$I_l = A \cdot \frac{b^2 \sin^2 \alpha}{12}, \quad \text{and} \quad \kappa_l = \sqrt{\frac{I}{A}} = \frac{b \sin \alpha}{2\sqrt{3}}.$$

We have also for the polar axis through the centre of mass C at right angles to the plane

$$I_z = \frac{A}{12}\left(h^2 + b^2 + \frac{h^2}{\tan^2 \alpha}\right).$$

(4) Moment of Inertia of the Area of a Hexagon.—We can divide the hexagon into six equilateral triangles of side b and area $A_1 = \frac{b^2\sqrt{3}}{4}$.



Take an axis YY in the plane of the area through the centre of mass perpendicular to the sides.

For the triangle ABD we have from page 273, since $d_1 = -\frac{b}{2}$, $d_2 = +\frac{b}{2}$, the moment of inertia $\frac{A_1 b^2}{24}$. For the triangle ABE we have, since $d_1 = \frac{b}{2}$, $d_2 = b$, the moment of the inertia is $\frac{7A_1b^2}{24}$. For the total moment of inertia with reference to YY we have then

$$I_y = \frac{7A_1b^2}{6} + \frac{A_1b^2}{12} = \frac{15A_1b^2}{12},$$

or, since $A = 6A_1$,

$$I_y = \frac{5Ab^2}{24}, \quad \text{and} \quad \kappa_y = \sqrt{\frac{I}{A}} = \frac{b\sqrt{5}}{2\sqrt{6}}.$$

If we take the axis XX through the centre of mass, we have, from page 273, for the moment of inertia of the triangle ABE, $\frac{A_1b^2}{8}$, and for the moment of inertia of the triangle ABD, $\frac{3A_1b^2}{8}$. The total moment of inertia with reference to XX is then

$$I_x = \frac{A_1b^2}{2} + \frac{3A_1b^2}{4} = \frac{5A_1b^2}{4},$$

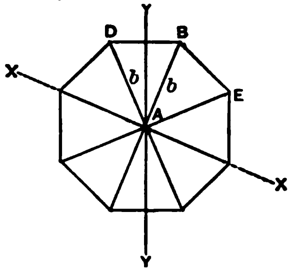
or, since $A = 6A_1$,

$$I_x = \frac{5Ab^2}{24}, \quad \text{and} \quad \kappa_x = \sqrt{\frac{I}{A}} = b \cdot \frac{\sqrt{5}}{2\sqrt{6}}.$$

For the polar axis through the centre of mass, perpendicular to the plane,

$$I_z = \frac{5Ab^2}{12}, \quad \text{and} \quad \kappa_z = \sqrt{\frac{I}{A}} = \frac{b\sqrt{5}}{2\sqrt{3}}.$$

(5) Moment of Inertia of the Area of an Octagon.—We can divide the octagon into eight isosceles triangles.



We find the moment of inertia with reference to an axis YY in the plane of the area, through the centre of mass perpendicular to the sides,

$$I_y = \frac{Ab^2}{24}(\sqrt{2+4}),$$

and

$$\kappa_y = \sqrt{\frac{I}{A}} = b\,\frac{\sqrt{5.414}}{2\sqrt{6}}.$$

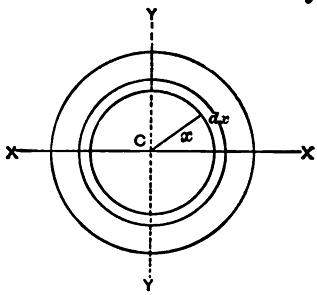
For the polar axis through the centre of mass, perpendicular to the plane, we have then

$$I_z = \frac{Ab^2}{12}(\sqrt{2} + 4), \quad \text{and} \quad \kappa_p = \sqrt{\frac{I}{A}} = b \, . \, \frac{\sqrt{5.414}}{2\sqrt{3}}.$$

For the axis XX in the plane of the area, through the centre of mass, coinciding with the sides, we obtain

$$I_x = \frac{Ab^2}{24}(\sqrt{2} + 4), \quad \text{and} \quad \kappa_x = \sqrt{\frac{I}{A}} = b\,\frac{\sqrt{5.414}}{2\sqrt{6}}.$$

(6) Moment of Inertia of the Area of a Circle.—The area of any circular strip of radius x and thickness dx is $2\pi x dx$. Its moment of inertia with reference to the polar axis through the centre of mass is then $2\pi x^3 dx$. The polar moment of inertia is then, since $\pi r^2 = A$ = the area,



$$I_z = \int_0^r 2\pi x^3 dx = \frac{\pi r^4}{2} = A \, . \, \frac{r^2}{2},$$

and

$$\kappa_z = \sqrt{\frac{I}{A}} = \frac{r}{\sqrt{2}}.$$

The moment of inertia with reference to any axis in the plane through the centre of mass, as XX or YY, is evidently the same, and, since $I_x + I_y = 2I = I_z$, we have for any axis in the plane through the centre of mass

$$I = \frac{\pi r^4}{4} = A \, . \, \frac{r^2}{4}, \quad \text{and} \quad \kappa = \sqrt{\frac{I}{A}} = \frac{r}{2}.$$

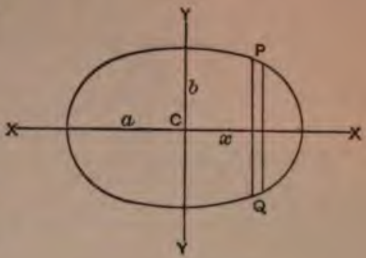
(7) Moment of Inertia of the Area of a Circular Ring.—Let r_1 = the internal radius and r_2 the external radius, so that the area is $\pi(r_2^2 - r_1^2) = A$. Then in the preceding case we have simply to integrate between r_2 and r_1, and we have for the polar axis through the centre of mass

$$I_z = \int_{r_1}^{r_2} 2\pi x^3 dx = \frac{\pi(r_2^4 - r_1^4)}{2} = \frac{\pi(r_2^2 - r_1^2)(r_2^2 + r_1^2)}{2} = A \, . \, \frac{r_2^2 + r_1^2}{2},$$

and for any axis through the centre of mass in the plane of the area

$$I = A \, . \, \frac{r_2^2 + r_1^2}{4}.$$

(8) **Moment of Inertia of the Area of an Ellipse.**—Let $a =$ the semi-major and b the semi-minor axes, and take the origin at the centre of mass. Then



$$y = \frac{b}{a}\sqrt{a^2 - x^2},$$

and the area $A = \pi ab$. The area of a strip, as PQ, is $2ydx$, and its moment of inertia with reference to the axis YY in the plane through the centre of mass is $2yx^2dx$.

Hence the moment of inertia of the area with reference to YY is

$$I_y = \frac{2b}{a}\int_{-a}^{+a} x^2\sqrt{a^2 - x^2}\,.\,dx = \frac{\pi a^3 b}{4} = A\,.\,\frac{A^2}{4}, \quad \text{or} \quad \kappa_y = \sqrt{\frac{I}{A}} = \frac{a}{2}.$$

In the same way we have for the moment of inertia with reference to the axis XX in the plane through the centre of mass

$$I_x = \frac{\pi b^3 a}{4} = A\,.\,\frac{b^2}{4}, \quad \text{or} \quad \kappa_x = \sqrt{\frac{I}{A}} = \frac{b}{2}.$$

The moment of inertia with reference to the polar axis through the centre of mass at right angles to the plane is then

$$I_z = A\,.\,\frac{a^2 + b^2}{4}, \quad \text{or} \quad \kappa_z = \frac{\sqrt{a^2 + b^2}}{2}.$$

Rule for Moment of Inertia of the Area of a Rectangle, Parallelogram, Circle or Ellipse with Reference to an Axis of Symmetry through the Centre of Mass.—The preceding is sufficient to illustrate how the moment of inertia of any area may be found. The use made of the moment of inertia will appear later. The various rolling mills furnish their customers with extensive Tables giving the moment of inertia of the cross-section of the different sizes and shapes of iron and steel beams rolled by them.* It is therefore unnecessary to multiply illustrations here.

We give here a simple rule which will enable the student to find at once the moment of inertia *with reference to an axis of symmetry through the centre of mass*, for the area of the rectangle, parallelogram, circle or ellipse. This rule is as follows:

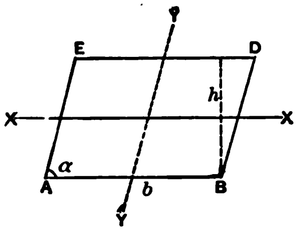
Axis of symmetry in plane of area through centre of mass: $\Big\}\; I = \text{area} \times \dfrac{\text{square of the other perpendicular semi-axis}}{3 \text{ or } 4}$;

Polar axis through centre of mass: $\Big\}\; I = \text{area} \times \dfrac{\text{sum of squares of two perpendicular semi-axes of symmetry}}{3 \text{ or } 4}$.

The denominator 3 or 4 is taken according as the area is a parallelogram or an ellipse. The rectangle and circle are special cases of parallelogram and ellipse.

* A most extensive collection is the "*Pocket Companion*" *of Carnegie, Phipps & Co., Pittsburgh, Pa.*

(1) Parallelogram and Rectangle.—Thus for the parallelogram $ABDE$ of base b and height h, we have for the axis of symmetry XX through the centre of mass C



$$I_x = \frac{A}{3}\left(\frac{h}{2}\right)^2 = A \cdot \frac{h^2}{12},$$

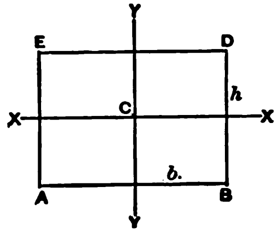
and

$$\kappa_x = \sqrt{\frac{I}{A}} = \frac{h}{2\sqrt{3}}.$$

For the axis of symmetry YY we have

$$I_y = \frac{A}{3}\left(\frac{b\sin\alpha}{2}\right)^2 = A\frac{b^2\sin^2\alpha}{12}, \quad \text{and} \quad \kappa_y = \sqrt{\frac{I}{A}} = \frac{b\sin\alpha}{2\sqrt{3}}.$$

For the rectangle we have



$$I_x = \frac{A}{3}\left(\frac{h}{2}\right)^2 = A\frac{h^2}{12},$$

and $$\kappa_x = \sqrt{\frac{I}{A}} = \frac{h}{2\sqrt{3}};$$

$$I_y = \frac{A}{3}\left(\frac{b}{2}\right)^2 = A\frac{b^2}{12},$$

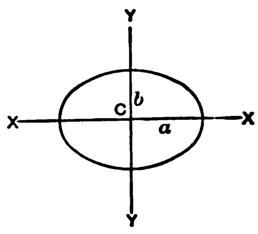
and $$\kappa_y = \sqrt{\frac{I}{A}} = \frac{b}{2\sqrt{3}};$$

and for the polar axis through C,

$$I_z = \frac{A}{3}\left(\frac{h^2}{4} + \frac{b^2}{4}\right) = A \cdot \frac{h^2 + b^2}{12} = A\frac{d^2}{12}, \quad \kappa_z = \sqrt{\frac{I}{A}} = \frac{d}{2\sqrt{3}},$$

where d is the diagonal of the rectangle. These are the same results as already obtained pages 272 and 274.

(2) Ellipse and Circle.—For the ellipse let a = the semi-major and b the semi-minor axis.



Then for the axis of symmetry XX through the centre of mass C we have

$$I_x = A\frac{b^2}{4}, \quad \text{and} \quad \kappa_x = \sqrt{\frac{I}{A}} = \frac{b}{2}.$$

For the axis of symmetry YY we have

$$I_y = A\frac{a^2}{4}, \quad \text{and} \quad \kappa_y = \sqrt{\frac{I}{A}} = \frac{a}{2}.$$

For the polar axis through C

$$I_z = A \cdot \frac{a^2 + b^2}{4}, \quad \text{and} \quad \kappa_z = \sqrt{\frac{I}{A}} = \frac{\sqrt{a^2 + b^2}}{2}.$$

For the circle $a = b = r$ = radius, and we have

$$I_x = I_y = A\frac{r^2}{4}, \quad \text{and} \quad I_z = A\frac{r^2}{2}.$$

These are the same results as already obtained pages 277 and 277.

Stress and Strain.—When a force is distributed over some definite portion of the surface of a body, we call it **external stress**, or stress *on* a body. A force between two particles or portions of a body is called **internal stress**, or stress *in* a body. External stress causes change of shape or volume of a body. Internal stress opposes such change of shape or volume.

We distinguish three kinds of simple stress:

Tensile stress, tending to pull the particles of a body apart in parallel straight lines, or resisting such separation.

Compressive stress, tending to push the particles of a body together in parallel straight lines, or resisting such approach.

Shearing stress, tending to cut a body across or to make the particles move past one another in parallel lines at right angles to the line joining the particles, or resisting such action; as in cutting with a pair of shears or in punching a plate.

We measure stress, then, whether external or internal, in pounds per square inch or per square foot.

The *change of distance* between two particles of a body in a direction opposite to coexisting internal stress between those particles is called **strain**. We distinguish strain according to the character of the internal stress to which it is opposite in direction, as tensile, compressive or shearing stress. We measure strain, then, in feet or inches.

It will be observed that when there is no coexisting internal stress, or if the internal stress is not opposite in direction to the change of distance, there is no strain. Internal stress and strain must coexist and be opposite in direction.

Thus when a spring is compressed the external and internal stresses balance, and the strain is the distance through which the end of the spring has been moved, counting from the unstrained position or the neutral point, where there is no external or internal stress. Now let the external stress be removed or the spring released. Then during the first portion of the expansion the internal stress acts in the same direction as the expansion, and this expansion cannot then be considered as a strain. The spring is not strained by such expansion; on the contrary the original strain is diminished.

But after the end of the spring passes the neutral point, if the spring still continues to expand, the internal stress is opposite in direction to the expansion, and any expansion beyond this point is a strain. The spring is strained by such expansion. In this case, then, we have strain without any external stress.

Experimental Laws.—Experiments made upon materials have established the following laws:

1. When a small stress, either tensile or compressive or shearing, is applied to a body, a small corresponding tensile, compressive or shearing strain is produced, and on the removal of the stress the body returns to its original dimensions.

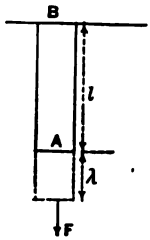
When the stress, either tensile or compressive or shearing, exceeds a certain amount, which varies according to the character of the stress and the material, the body on removal of the stress does not return to its original dimensions. The portion of the strain which remains permanent is called the **set**. The unit stress for which set is first observed is called the **elastic limit** for tension, compression or shear.

3. So long as no set is observed, or so long as the unit stress is

less than the elastic limit, ***the strain is proportional to the stress which produces it.*** After set is observed, or when the unit stress is greater than the elastic limit, the strain increases more rapidly than the stress which produces it, until finally rupture occurs.

4. A suddenly applied stress or shock is more injurious than a steady stress or a stress gradually applied.

Determination of the Elastic Limit.—Let a bar AB of uniform cross-section A have an external stress or force F applied to it which elongates, compresses or shears the bar. In the figure we suppose elongation. As the bar then elongates, internal stress acts in a direction opposite to the elongation. The elongation is then a strain. Denote this strain by λ and let the original length of the bar be l. Let s be the strain per unit of length. Then we have



$$s = \frac{\lambda}{l}. \qquad (1)$$

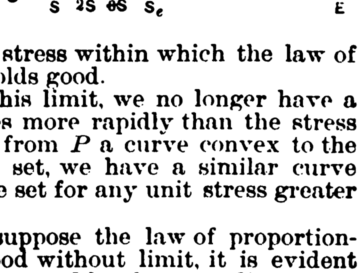
If the external stress or force F is applied in the axis of the bar, the internal unit stress or stress per square unit of cross-section is

$$S = \frac{F}{A}. \qquad (2)$$

Now according to the laws just stated, so long as the unit stress is less than the elastic limit S_e, *the strain is proportional to the applied stress which produces it*, and no set will be observed upon removal of the stress.

If then we double the external stress F, we shall observe a double strain 2λ, and so on.

It is evident that if we lay off the unit stresses $S = \frac{F}{A}$, $2S = \frac{2F}{A}$, $3S = \frac{3F}{A}$, etc., to scale along a horizontal line, and lay off the corresponding observed strains λ, 2λ, 3λ, etc., as ordinates, we shall obtain, so long as the unit stress S does not exceed the elastic limit S_e, *a straight line OP.*



By thus carefully plotting the results of experiment, whether of compression, tension or shear, we can detect the point P at which deviation from the straight line occurs. The corresponding unit stress S_e is the elastic limit for tension, compression or shear.

The elastic limit is then the unit stress within which the law of proportionality of strain to stress holds good.

When the unit stress exceeds this limit, we no longer have a straight line, but the strain increases more rapidly than the stress until rupture occurs, and we have from P a curve convex to the horizontal. Also if we observe the set, we have a similar curve S_eQ, the ordinates to which give the set for any unit stress greater than S_e.

Coefficient of Elasticity.—If we suppose the law of proportionality of strain to stress to hold good without limit, it is evident that the results of experiment represented by the preceding figure

will enable us to calculate the unit stress which would cause a strain equal to the original length l. This unit stress is called the **coefficient of elasticity**. We denote it by E.

The coefficient of elasticity, then, *is that unit stress which would cause a strain equal to the original length provided the law of proportionality of strain to stress were to hold good without limit.*

We can easily compute it from the preceding figure. Thus let the straight line OP be produced indefinitely and let the strain $EB = l =$ the original length. Then OE gives the coefficient of elasticity E, and we have by similar triangles

$$S : \lambda :: E : l, \quad \text{or} \quad E = \frac{lS}{\lambda} = \frac{lF}{\lambda A}, \quad . \quad . \quad . \quad . \quad (1)$$

since the unit stress $S = \frac{F}{A}$, where F is the applied stress and A is the area over which it is distributed.

Since the strain per unit of length $s = \frac{\lambda}{l}$, we also have

$$E = \frac{S}{s}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or, *the coefficient of elasticity is the ratio of the unit stress to the unit strain.*

From (1) we can determine E by experiment for any given material. When E is thus known we can find in any case the strain caused by any unit stress *within the elastic limit*, by the equation

$$\lambda = \frac{lS}{E} = \frac{lF}{EA}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Inversely, the stress F corresponding to the strain λ is given *within the elastic limit* by

$$F = \frac{\lambda}{l} AE = sAE. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

These formulas apply either to extension, compression or shear.

Work and Coefficient of Resilience.—If the unit stress S does not exceed the elastic limit S_e, we see from the figure page 280 that since OP is a straight line, the work done per unit of area is equal to the unit stress multiplied by the mean strain which is $\frac{\lambda}{2}$. We have then for the work per unit of area done by the unit stress S in causing the strain λ

$$\frac{W}{A} = \frac{1}{2} S\lambda,$$

or, since the total stress $F = SA$,

$$W = \frac{1}{2} F\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or the work of the stress F in causing the strain λ is *one half the product of the stress and strain within the elastic limit.*

At the elastic limit we have from equation (3),

$$\lambda = \frac{lS_e}{E}, \quad \text{and} \quad F = S_e A.$$

Hence the work done in straining the body to the elastic limit is

$$W = \frac{S_e^2}{2E} \cdot Al = \frac{S_e^2}{2E} \cdot V, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where V is the volume of the body, or $V = Al$. Since at the elastic limit there is no set, this is the work which the body can do in returning to its original dimensions. It is therefore called the **work of resilience**. The coefficient $\frac{S_2^2}{2E}$, or the work per unit of volume, is called the **coefficient of resilience**.

The work of resilience is then *the work which a body can do in returning to its original dimensions when it has been strained up to the elastic limit.*

The coefficient of resilience is *the work per unit of volume* done by the body under such circumstances.

The work of resilience measures the ability of the material to withstand shock or the suddenly applied stress produced by a moving body. To bring such a body to rest requires work. If this work is not greater than the work of resilience, the elastic limit is not exceeded.

From the Table of Average Properties of Materials given on page 290 we can compute the following average values of the coefficient of resilience:

	Coefficient of Resilience.
Timber..........	3 inch-pounds per cubic inch.
Cast iron........	1.2 " " " " "
Wrought iron...	12.5 " " " " "
Steel............	26.6 " " " " "

We see from the figure page 280 that we cannot express the work done in straining a body to the breaking point by a formula, because the law of the relation of stress to strain beyond the elastic limit is unknown. Moreover, such work could not be properly termed work of *resilience*, since it can not be performed by the body when the stress is removed. The body if strained beyond the elastic limit does not return to its original length. Work of resilienc ethen is a measure of capacity to resist shock *within the elastic limit only*.

Conditions of Equilibrium of a Deflected Beam.—A bar of any cross-section, constant or variable, whose length is great compared to its other dimensions and which is acted upon by forces at right angles to its length is called a **beam**. A **cantilever** beam is fixed at one end and free at the other. A beam in general rests upon supports at both ends. When a beam rests on more than two supports it is said to be **continuous**.

Reactions of the Supports.—The supports of a beam exert pressures called **reactions**. When a beam resting upon supports and acted upon by external loads or forces either concentrated or distributed, is at rest, we must have for equilibrium, since the loads and reactions may be considered as co-planar (page 99):

1st. The algebraic sum of all the vertical forces $= 0$;

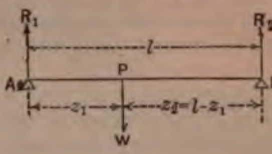
2d. The algebraic sum of all the horizontal forces $= 0$;

3d. The algebraic sum of the moments of all forces with reference to any point in the plane of the forces $= 0$.

If the 1st condition is complied with, there is no motion up or down. If the 2d is complied with, there is no motion right or left. If the 3d is complied with, there is no rotation.

In taking the algebraic sums, forces upwards or to the right are positive, downwards or to the left are negative. Moments which tend to cause counter-clockwise rotation are positive, clockwise rotation negative.

Thus suppose we have a horizontal beam AB of length l, resting on the supports A and B in a horizontal line, and loaded with a weight W at a distance z_1 from the left end. Then there are no horizontal forces and condition (2) is satisfied.



In order that condition (1) may be satisfied, let R_1 and R_2 be the reactions. Then

$$R_1 + R_2 - W = 0.$$

Take B as a point of moments. Then in order that condition (3) may be satisfied, we must have

$$-R_1 l + W(l - z_1) = 0.$$

From these two equations, if we put $l - z_1 = z_2$, we obtain

$$R_1 = \frac{W(l - z_1)}{l} = \frac{Wz_2}{l}, \qquad R_2 = \frac{Wz_1}{l},$$

or the reactions are positive and therefore act upwards and are inversely as the segments z_1, z_2 into which the span l is divided by the load W.

If the load is w per unit of length, uniformly distributed, then the entire load is wl, and we can consider this entire load as a single force acting at the centre of mass of the loading, or at the distance $\frac{l}{2}$ from each end.

Since there are no horizontal forces, condition (2) is satisfied. In order to satisfy condition (1), we must have

$$+ R_1 + R_2 - wl = 0.$$

Taking B as a point of moments, in order to satisfy condition (3) we have

$$-R_1 l + wl \times \frac{l}{2} = 0.$$

From these two equations we obtain $R_1 = R_2 = \frac{wl}{2}$, or the reaction at each support is positive and therefore upwards and equal to one half the total distributed load.

We can find in similar manner the reactions at the supports in any case. (For determination of reactions in general, see page 100.)

Shearing Force and Shearing Stress.—The algebraic sum of the components parallel to a section at any point, of all the external forces *on the left of* that section, we call the **shearing force** of that section.

It is the force which tends to make the section slide upon the next consecutive section on the right.

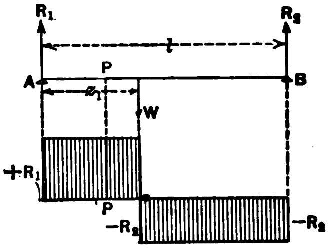
It is resisted by the **shearing stress** or resistance of the section to sliding. In the case of a beam acted upon by vertical forces, the algebraic sum of all the vertical forces *on the left* of any vertical

cross-section is the *vertical* shearing force at that cross-section. If x is the distance of the cross-section from the left origin, we denote it by V_x. If then S_{ws} is the allowable or working unit shearing stress of the material and A is the area of vertical cross-section of the beam at any point, the safe resistance to shear or the shearing stress of the beam at that point is $S_{ws}A$. This must be equal and opposite to the vertical shearing force V_x. We must have then for safety as regards shearing at any point

$$S_{ws}A \overset{=}{>} - V_x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If V_x is positive or upwards for horizontal beam, $S_{ws}A$ is negative or downwards, and inversely.

Thus for a horizontal beam of length l, resting on the supports A and B and loaded with the weight W at a distance z_1 from the left end, the left reaction is, as we have just seen, $R_1 = \dfrac{W(l - z_1)}{l}$.



This then, according to definition, is the shearing force V_x for any point P between the load W and the left end A.

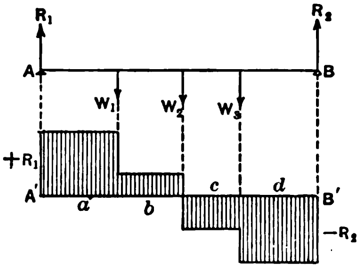
For any point between the load W and the right end B the shearing force is

$$V_x = + R_1 - W = - \frac{Wz_1}{l} = - R_2.$$

The shaded area in the figure gives the shear at any point.

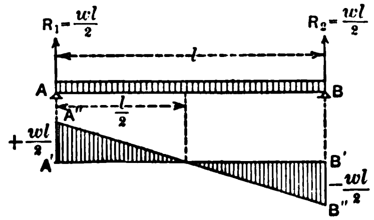
If we have several loads W_1, W_2, W_3, etc., then for any point a between the left support and W_1 we have $V_x = R_1$. For any point b between W_1 and W_2 we have $V_x = R_1 - W_1$. For any point c between W_2 and W_3 we have $V_x = R_1 - W_1 - W_2$. For any point d between W_3 and the right end

$$V_x = R_1 - W_1 - W_2 - W_3 = - R_2.$$



The shaded area gives the vertical shear at any point.

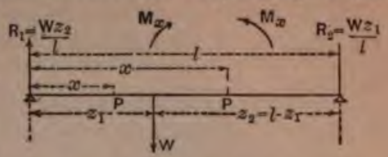
If we have a load w per unit of length uniformly distributed, we have at any point distant x from the left end



$$V_x = \frac{wl}{2} - wx,$$

which is the equation to a straight line $A''B''$. The ordinate at any point a to this line is the shear at that point. The shear at the centre is evidently zero. At the left end A it is $+ \dfrac{wl}{2}$, and at the right end B it is $- \dfrac{wl}{2}$.

Bending Moment.—In the case of the horizontal beam with a concentrated load W at the distance z_1 from the left end, let M_x be the algebraic sum of the moments with reference to any point P distant x from the left end, of all the external forces *between that point and either end.*



This moment tends to turn that portion of the beam on the left or right of any point about that point, or to cause bending. It is therefore called the **bending moment.**

We have evidently two cases: when x is less than z_1 or when the point P is on the left of W, and when x is greater than z_1 or when the point P is on the right of W.

Let us take the algebraic sum of the moments of all the forces *on the left* of the point P. Then we have for the bending moment at the point P for the case represented by the figure,

when $x < z_1$, $$M_x = -R_1x = -\frac{Wz_2x}{l} = -\frac{W(l - z_1)x}{l};$$

when $x > z_1$,

$$M_x = -R_1x + W(x - z_1) = -\frac{Wz_1(l - x)}{l} = -R_2(l - x).$$

The minus sign shows that the forces on the left of any point P in the case represented by the figure tend to cause clockwise rotation of the left-hand portion AP of the beam about that point.

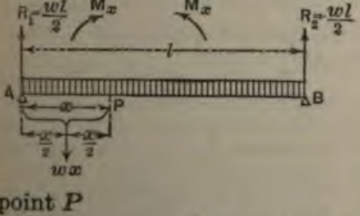
If we take the algebraic sum of the moments of all the forces *on the right* of the point P, we evidently have for the bending moment at the point P,

when $x < z_1$, $M_x = +R_1x$; when $x > z_1$, $M_x = +R_2(l - x)$.

The plus sign shows that the forces on the right of any point P in the case represented by the figure tend to cause counter-clockwise rotation of the right-hand portion BP of the beam about that point.

In general, since the beam is in equilibrium, *the bending moment due to all the forces on one side of any point is always equal in magnitude and opposite in direction to the bending moment due to all the forces on the other side of that point.*

In the case, again, of the horizontal beam with the load w per unit of length uniformly distributed, the load over any distance x from the left end is wx, and we can take this load as acting at its centre of mass, or at a distance $\frac{x}{2}$ from the left end and from P.



If we take the algebraic sum of the moments of all the forces *on the left* of the point P, we have for the bending moment at the point P

$$M_x = -R_1x + wx \times \frac{x}{2} = -\frac{wx}{2}(l - x).$$

Here again the minus sign shows that the forces on the left of

any point P tend to cause clockwise rotation of the left-hand portion AP of the beam about that point.

If we take the algebraic sum of the moments of all the forces *on the right* of P, we obtain $M_x = +\frac{wx}{2}(l - x)$, or counter-clockwise rotation.

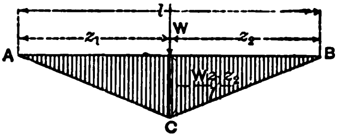
We see from the preceding illustrations how to find the bending moment M_x in any given case at any point P.

Although the beam bends under the action of the external forces, the deflection in all practical cases is always very small in comparison to the length.

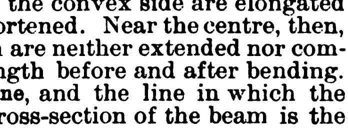
We therefore always consider the beam as straight in finding the reactions and bending moment; that is, *we assume the deflection as very small in comparison with the length.*

Graphic Representation of the Bending Moment.—The graphic method of page 148 can be used to determine the bending moment at any point of a beam.

For a beam with a single concentrated load we see at once from the preceding Article that the moment at the load is greatest and equal to $-\frac{Wz_1z_2}{l}$. The moment at each end is zero, and the ordinate at any point to the lines AC, BC gives the bending moment at that point.



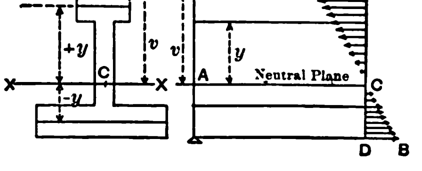
For a load w per unit of length uniformly distributed, the bending moment $M_x = -\frac{wx}{2}(l - x)$ is the equation of a parabola whose maximum ordinate at the centre of the span is $\frac{wl^2}{8}$. The ordinate at any point to this parabola gives the bending moment at that point.



Neutral Axis.—We consider a beam to be made up of an indefinitely great number of horizontal or parallel fibres of indefinitely small area of cross-section, placed side by side.

When a beam bends, the fibres on the convex side are elongated and those on the concave side are shortened. Near the centre, then, we must have a plane of fibres which are neither extended nor compressed, but remain of the same length before and after bending. This plane is called the **neutral plane**, and the line in which the neutral plane cuts the plane of any cross-section of the beam is the **neutral axis** for that cross-section.

Thus in the figure AC represents the neutral plane and XX the neutral axis.



Position of the Neutral Axis.—We assume that any cross-section, as DD, which is plane before flexure, *remains plane after flexure.* Thus let the plane DD before flexure be represented by the plane BB after flexure. Then the strain of any fibre is proportional to its distance from the neutral axis.

We also assume that *the elastic limit is not exceeded.* Hence the stress in any fibre is proportional to the strain and therefore proportional to the distance of the fibre from the neutral axis.

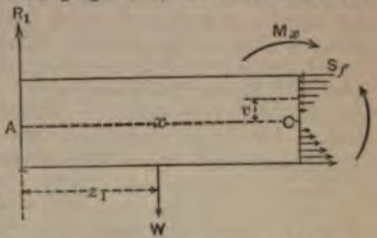
Let S_f be the unit stress *within the elastic limit* in the *extreme outer fibre of the cross-section*, or the fibre *most remote from the neutral axis*, and v its distance from the neutral axis. Let a be the cross-section of a fibre. Then the stress in the extreme outer fibre at the distance v is $S_f a$, and the stress in any other fibre at a distance y from the neutral axis is $\frac{y}{v} S_f a$. The sum of all the fibre stresses above and below the neutral axis is then

$$\Sigma \frac{y}{v} S_f a = \frac{S_f}{v} \Sigma ay.$$

But since the beam is in equilibrium and all the external forces are vertical, the sum of all the horizontal fibre stresses in any cross-section must be zero. We must have then $\Sigma ay = 0$, or *the neutral axis must pass through the centre of mass of the cross-section* (page 17).

The line AC passing through the centre of mass of every cross-section is the neutral axis of the beam.

Resisting Moment.—We have seen, page 285, how to find the *bending moment* M_x at any point of a beam distant x from the left end. The bending moment bends the beam or tends to cause the portion of the beam between the point and the left end to turn about that point.



In the figure take the point C on the neutral axis, distant x from the left end. Then, as we have seen (page 285), we have for the case represented, for the bending moment at any point of the cross-section at C,

$$M_x = -\frac{Wz_1(l-x)}{l}$$

if $x > z_1$. This moment is negative and hence the effect of the external forces R_1 and W on the left of C is to cause clockwise rotation of the portion AC of the beam about C.

But if the beam is in equilibrium, *the bending moment M_x must be balanced by the sum of the moments of the fibre stresses of the cross-section above and below C, with reference to C.*

Now any fibre stress of the cross-section, at a distance y from the neutral axis, is, as we have just seen, $\frac{y}{v} S_f a$, where a is the cross-section of the fibre and S_f the unit stress within the elastic limit in the most remote fibre of the cross-section at the distance v from the neutral axis. The moment of any fibre stress at the distance y from the neutral axis is then $\frac{S_f}{v} ay^2$, and the sum of all the fibre-stress moments of the cross section with reference to the neutral axis is $\frac{S_f}{v} \Sigma ay^2$.

But (page 271) Σay^2 is the moment of inertia I of the cross-sec-

tion with reference to the neutral axis. Hence the sum of the moments of all the fibre stresses of the cross-section with reference to the neutral axis at C is

$$\frac{S_f I}{v}.$$

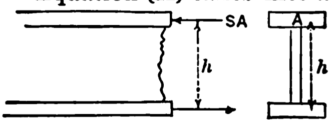
We call this the **resisting moment**, because it resists the action of the bending moment M_x and thus prevents the portion of the beam AC from turning about the neutral axis at C under the action of the external forces on the portion AC. The bending moment M_x is therefore always equal in magnitude and opposite in direction to the resisting moment. If we consider always the fibres belonging to that portion of the beam *on the left of the cross-section*, then the resisting moment of these fibres is always opposite in direction to the bending moment of all external forces on the left and in the same direction as the bending moment of all external forces on the right. We have then

$$\frac{S_f I}{v} = \mp M_x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II)}$$

where we take the minus sign if we take M_x for all external forces *on the left*, and the plus sign if we take M_x for all external forces *on the right*, the resisting moment being *always* that due to the fibre stresses of the left-hand portion. If this latter moment comes out minus, it indicates then compression in the bottom fibres; if plus, tension in the bottom fibres.

By the use of (II) we can find, in any given case, the load which a beam will carry for any given value of S_f *within the elastic limit.* We can also determine the shape of the beam for uniform strength, that is, for S_f the same at every cross-section.

Equation (II) takes into account the fibre stresses of the entire cross-section whatever its shape. If a beam consists of two flanges and a web, it is sometimes customary to disregard the web. In such case, if h is the *effective height* or distance from centre to centre of flanges, and A is the area of one flange at any point, and S the unit stress, we have, taking moments about the centre of the other flange,



$$SAh = \mp M_x.$$

Coefficient of Rupture.—In all the preceding discussion of the equilibrium of a beam we have assumed—

1st. That the deflection is very small compared to the length.

2d. That any cross-section plane before flexure remains plane after.

3d. That the elastic limit is not exceeded.

When a beam is loaded to the *point of rupture*, the third assumption is violated. The strain is then no longer directly as the distance from the neutral axis, and the second assumption no longer holds. Also, the first is often not allowable.

We can therefore properly apply equation (II) only when the elastic limit is not exceeded.

Now when a beam is loaded to the point of rupture, we assume an equation of the *same form* as (II), and write

$$\frac{S_r I}{v} = \mp M_r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III)}$$

where M_r is the bending moment at the cross-section where rupture occurs, or the **dangerous cross-section**, I is the moment of inertia with reference to the neutral axis of that cross-section, and S_r is the unit stress in the most remote fibre of that cross-section at the distance v from the neutral axis *where rupture occurs.*

When the cross-section of the beam is constant, I and v are constant, and we see from (II) that the outer fibre stress S_f is greatest at the point where the bending moment M_x is greatest. The dangerous cross-section for a beam of constant cross-section is then the one for which the bending moment is a maximum.

The value of S_r determined from equation (III) by experiments made at the breaking point is called the **coefficient of rupture**.

Let S_t be the unit stress of *direct* tension and S_c the unit stress of *direct* compression which ruptures a bar. We call S_t the **ultimate tensile strength**, and S_c the **ultimate compressive strength.** The ultimate compressive strengths of tension and compression are not in general equal. Thus for timber (Table page 290) the tensile strength is the greater, while for cast iron the compressive strength is the greater.

If equation (II) held good beyond the elastic limit, we should expect to find S_r in (III) equal to the least ultimate strength of the material, either tension or compression as the case may be. But as a matter of fact S_r is always found by experiment to lie nearly midway between S_t and S_c when they are different.

Experiments upon the value of S_r are not numerous; and when in any case the value of S_r is not known, but S^t and S_c are known, we can find an approximate value for S_r by taking the mean value of S_t and S_c, or putting $S_r = \dfrac{S_t + S_c}{2}$.

By the use of (III), then, we can estimate more or less accurately the breaking weight of a beam.

Table of Properties of Materials. — We give in the following Table average values of the ultimate compressive strength S_c, the ultimate tensile strength S_t, the coefficient of rupture S_r, the elastic limit S_e and the ultimate strength S_u—all in pounds per square inch. We also give the coefficient of elasticity E in pounds per square inch as determined by experiments in direct compression, tension and shear. Also the density δ or mass of a cubic foot of material in pounds.

All these values are averages and liable to great variations for different grades and qualities of materials. Thus, for instance, timber varies in its qualities according to kind, and each kind varies according to soil, climate, season when cut, method and duration of seasoning, direction of fibres with reference to stress, form and size of test specimen, etc. So, also, iron and steel vary according to quality, process of manufacture, whether in bars, plates or wire, etc. Such average values as we give, then, can only be used in preliminary computations. In actual cases of investigation and design, special experiments must be made with the materials actually used.

As to density or mass per cubic foot, a rule which should be noted by the student is *that a bar of wrought-iron one square inch in cross-section and one yard long* (*or* 36 *cubic inches*) *weighs ten pounds.* Thus the weight per foot in pounds of a bar of uniform cross-section is at once given by multiplying the area of cross-section in square inches by 10 and dividing by 3. Inversely, if the weight per foot in pounds is given, multiply by 3 and divide by 10 for the area of cross-section in square inches.

Steel is about two per cent heavier and cast iron six per cent lighter than wrought iron.

Stone is about one third, brick one fourth, timber one twelfth the weight of wrought iron.

When a test specimen is ruptured by direct tension, it elongates rapidly after the elastic limit is reached, and the area of cross-section is in general greatly reduced. The ultimate elongation, taken in connection with the reduction of area, indicates the *ductility* of the material.

Thus a material which has a high ultimate strength but shows little elongation and reduction of area is *brittle*. We have therefore given in the Table the average value of the ultimate elongation per unit of original length $s = \frac{\lambda}{l}$.

TABLE OF AVERAGE PROPERTIES OF MATERIALS.

	Ultimate Compressive Strength. S_c	Ultimate Tensile Strength. S_t	Ultimate Elongation $s = \frac{\lambda}{l}$	Coefficient of Rupture. S_r	Elastic Limit. S_e
	lbs. per sq. in.	lbs. per sq. in.	in. per lin. in.	lbs. per sq. in.	lbs. per sq. in.
Timber........	8000	10000	0.015	9000	3000
Brick..........	2500	....	....	....	....
Stone..........	6000	....	... .	2000	....
Cast iron.......	90000	20000	0.005	35000	6000 tension; 60000 compression
Wrought iron..	55000	55000	0.15	55000	25000
Steel (structural)	150000	100000	0.10	120000	40000

	Ultimate Shearing Strength. S_u	Coefficient of Elasticity E	Density δ
	lbs. per sq. in.	lbs. per sq. in.	lb. p. cu. ft.
Timber........	600 longitudinal; 3000 transverse	1500000 tens. or compress.; 400000 shear	40
Brick..........	....	....	125
Stone..........	....	6000000 compression	160
Cast iron.......	20000	15000000 tens. or compress.; 6000000 shear	450
Wrought iron .	50000	25000000 tens. or compress.; 15000000 shear	480
Steel (structural)	70000	30000000 tens. or compress.	490

Factor of Safety and Working Stress.—The ratio in any case of the ultimate strength to the actual working unit stress is called the **factor of safety**. Thus if the ultimate stength or unit stress at the point of rupture in any case is denoted in general by S_u, and if S_w is the working unit stress, we have for the factor of safety in that case

$$n = \frac{S_u}{S_w}, \quad \text{or} \quad nS_w = S_u.$$

The factor of safety, then, is a number which tells how many times the actual unit stress are necessary to produce rupture.

The safe or working unit stress is then found by dividing the ultimate strength by the proper factor of safety. It should always be well within the elastic limit. If then the elastic limit is known, the working stress can be chosen with reference to it. This is the best and most rational method of determining the working unit stress. But it is in many cases difficult to determine the elastic limit, while the ultimate strength is more readily and definitely determined and in general better known. Hence the employment of a factor of safety in connection with the ultimate strength.

The following Table gives the average values of the factors of safety usually adopted. These values are not to be used arbitrarily, but in the light of judgment and experience. In any important engineering structure special experiments upon the materials actually used should be made in order to determine their properties as to coefficient of elasticity, elastic limit, ultimate strength, etc., and materials not coming up to a specified standard rejected. From such experiments the working stress can be decided in view of the actual qualities of the material. The average values in the Table can, however, be used for preliminary estimates.

TABLE OF AVERAGE FACTORS OF SAFETY.

Material.	For Steady Stress (Buildings).	For Varying Stress (Bridges, etc.).	For Shocks (Machines).
Timber	8	10	15
Brick and stone	15	25	30
Cast iron	6	10	15
Wrought iron	4	6	10
Steel (structural)	5	7	10

In order, then, to find the working unit stress S_w in any case, we divide the ultimate unit stress S_u by the factor of safety n, as given by the preceding Table. This gives us in any case a constant working unit stress $S_w = \frac{S_u}{n}$. For average values we have then the following Table for working unit stress, which may be used for preliminary estimates.

TABLE OF WORKING UNIT STRESS S_w IN POUNDS PER SQUARE INCH.

Material.	Steady Stress (Buildings). S_w			Varying Stress (Bridges, Roofs, etc.). S_w			Shocks (Machines, etc.). S_w		
	Tens.	Comp.	Shear.	Tens.	Comp.	Shear.	Tens.	Comp	Shear.
Timber	1300	1000	80 long. 400 trans.	1000	800	60 long. 300 trans.	700	600	40 long. 200 trans.
Brick	.....	170	....	.....	100	....	.....	80	....
Stone	.....	400	....	.....	240	....	.....	200	....
Cast iron	3300	15000	3300	2000	9000	2000	1300	6000	1300
Wrought iron	14000	14000	12500	9000	9000	9000	5500	5500	5000
Steel (structural)	20000	30000	14000	14000	21000	10000	10000	15000	7000

In order to determine the area of cross-section A for simple tension or compression or shear, we have then simply to divide the total stress by the working unit stress S_w. We have then, *when flexure is not to be apprehended*, for steady or varying stress or shocks,

$$A = \frac{\text{total stress}}{S_w}.$$

Sometimes we have *alternating stress*, i.e., sometimes tension and sometimes compression, as in the connecting-rod of a steam-engine. In such case it is a common practice, for the sake of security, to find the area of cross-section for each stress and take the sum. Thus, *if flexure is not to be apprehended*,

$$A = \frac{\text{total tensile stress}}{S_w} + \frac{\text{total compressive stress}}{S_w}.$$

When flexure is to be provided against, we must proceed as on page 361.

Variable Working Stress.—The fact that the working unit stress S_w, as determined in the preceding Article, is constant in any case is by many engineers considered objectionable.

The total unit stress can in general be divided into two portions. The one portion is a steady unit stress always existing, such as that due to weight or dead load. The other portion is a repeated unit stress such as that due to loads recurring at intervals.

Evidently, when the ratio of the steady stress to the total stress is great, we should be able to take a greater working unit stress than when it is small. Thus when the steady stress is equal to the total stress, there is no repeated stress at all and the working unit stress should have its greatest value. On the other hand, when the steady stress is zero, we have repeated stress only and the working stress should have its least value.

It is therefore customary to take for the working unit stress, *when flexure is not to be apprehended*, for *repeated stress*,

$$S_w = \frac{S_p}{n}\left(1 + \frac{S_u - S_p}{S_p} \cdot \frac{\text{steady stress}}{\text{total stress}}\right). \quad . \quad . \quad . \quad \text{(I)}$$

From equation (I) we see that when the steady stress is equal to the total stress, that is, when there is no repeated stress, we have $S_w = \frac{S_u}{n}$, where S_u is the ultimate strength and n the factor of safety, just as in the preceding Article.

But when the steady stress is zero, we have only repeated stress, and equation (I) gives us $S_w = \frac{S_p}{n}$. Hence S_p must be the ultimate strength for repeated stress. We call this the "*repetition strength.*"

In like manner, *when flexure is not to be apprehended*, we have for the working unit stress, for *alternating stress*,

$$S_w = \frac{S_p}{n}\left(1 - \frac{S_p - S_v}{S_p} \cdot \frac{\text{least of the two opposite stresses}}{\text{greatest of the two opposite stresses}}\right). \quad \text{(II)}$$

From equation (II) we see that when one of the two opposite stresses is zero we have $S_w = \frac{S_p}{n}$, as in the previous case for steady stress zero.

But when the two opposite stresses are equal we have $S_w = \frac{S_v}{n}$. Hence S_v must be the ultimate strength for equal alternating stresses. We call this the "*vibration strength*."

The difficulty and uncertainty of determining S_p and S_v by experiment, and the few experiments available, make the method of the preceding Article the most generally accepted.

The method of equations (I) and (II) of the present Article is, however, the most rational, and it is quite extensively used with certain assumed average values for S_u, S_p and S_v, as given in the following tabulation:

	$\frac{S_p}{n}$	$\frac{S_u - S_p}{S_p}$	$\frac{S_p - S_v}{S_p}$
Wood	400	2	$\frac{1}{2}$
Wrought iron............	7500	1	$\frac{1}{2}$
Cast iron..............	10000	$\frac{4}{3}$	$\frac{2}{5}$
Steel (structural)	17870	1	$\frac{7}{15}$

These values are for direct stress of tension or compression. For shear we take four fifths of S_w as determined above.

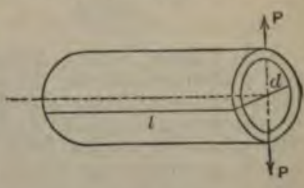
In order to determine the area of cross-section A, we have in all cases

$$A = \frac{\text{total maximum stress}}{S_w}.$$

When flexure is to be provided against we must proceed as on page 361.

Strength of Pipes and Cylinders. — Let p be the pressure per square inch on the interior surface of a pipe or cylinder due to the pressure of water or steam. It is a well-known principle of physics that the pressure of a fluid in any direction is equal to the pressure on a plane perpendicular to that direction.

Hence in the figure the pressure P, say in a vertical direction, is equal to the pressure on a horizontal plane ld, where l is the length and d is the interior diameter. We have then $P = pld$. If S_w is the safe working unit stress for the material for tension, and t is the thickness, we must have then



$$pld = 2tlS_w, \quad \text{or} \quad t = \frac{pd}{2S_w}. \quad (1)$$

Pipes come in commercial sizes, and the preceding formula enables us to select the nearest commercial size for given pressure, diameter and safe working unit stress.

If we consider the preceding figure as a closed cylinder, then the

pressure on the head is $p \times \frac{\pi d^2}{4}$, and the area of cross-section is πdt. We have then

$$p \times \frac{\pi d^2}{4} = \pi dtS_w, \quad \text{or} \quad t = \frac{pd}{4S_w}. \quad . \quad . \quad . \quad . \quad (2)$$

Hence the thickness to resist longitudinal rupture is twice that necessary to resist end rupture. For water pressure, if the head h is taken in feet, the pressure in pounds per square inch is $p = 0.434h$.

Riveted Joints.—In a riveted joint the resistance of the rivets due to shear should equal the tensile strength of the plates joined.

Kinds of Riveted Joints. — We may distinguish the following joints:

1st. *Simple "Lap" Joint, Single-riveted.*—Fig. 1 shows this joint front and side. The two plates overlie each other by an amount equal to the "*lap*" and are united by a single row of rivets. The distance p from centre to centre of a rivet is called the *pitch*. We denote the diameter of rivet by d and the thickness of plate by t.

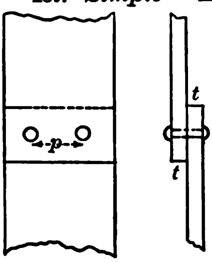


FIG. 1.

2d. *"Lap" Joint, Double-riveted.* — This joint is similar to the preceding, except two rows of rivets are used. In both cases the rivets are in *single shear*.

In all cases where more than one row of rivets is used the rivets are "*staggered*," or so spaced that those in one row come midway between those in the next, as shown in Fig. 2.

Lap joints are used in tension only.

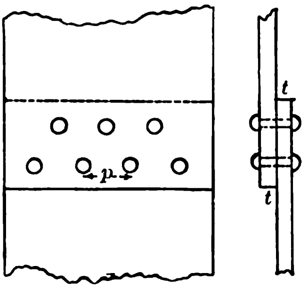


FIG. 2.

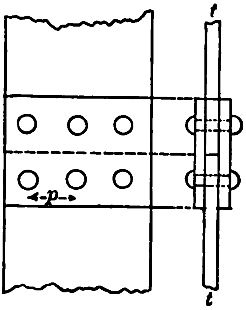


FIG. 3.

3d. *"Butt" Joint, Single-riveted, Two Cover-plates.*—Here the two plates are set end to end, making a "*butt*" *joint*, and a pair of "*cover-plates*" are placed on the back and front and riveted through by a single row of rivets on each side of the joint (Fig. 3). The plates in such a joint are in general not allowed to actually touch, and the entire stress, whether tensile or compressive, is therefore transmitted by the rivets. The thickness of the cover-plates should not be less than half the thickness of the plates joined, except when this rule would give a thickness less than $\frac{1}{4}$ inch. Owing to deterioration of the metal by the action of the weather, ***no plate is used in construction less than $\frac{1}{4}$ inch in thickness.*** Hence if the plates joined are less than $\frac{1}{2}$ inch, the cover-plates should be $\frac{1}{4}$ inch.

4th. "Butt" Joint, One Cover-plate, Single-riveted.—This is the same as the preceding except that one cover-plate only is used, of the same thickness as the plates themselves.

5th. Double-riveted "Butt" Joint, Two Cover-plates.—This joint is the same as case 3, except that we have two rows of rivets on each side of the joint.

The thickness of the cover-plates is determined by the same considerations as in case 3.

6th. "Butt" Joint, One Cover-plate, Double-riveted.—This is the same as the preceding case, except that there is only one cover-plate of the same thickness as the plates themselves.

7th. Chain Riveting.—When we have more than two rows of rivets on each side of a butt joint, the system is called chain riveting. Such a disposition becomes necessary when the requisite number of rivets is so great that they cannot be disposed in two rows without unduly weakening the plates.

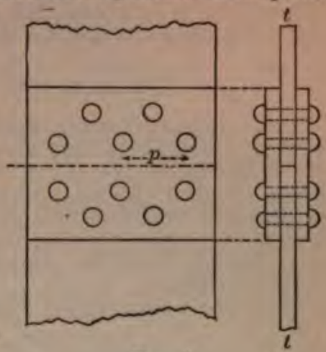


FIG. 4.

Theory and Practice of Riveting.—A rivet may fail by shearing across or by being crushed. The plate may fail by rupture between the rivets or by tearing through of the rivets at the edge of plate. The rivets should be so proportioned and spaced that the strength for any method of failure may be equal and the plates weakened as little as possible.

Notation.—Let S_w be the working unit stress of the plates, either compression or tension, S_{wc} the working unit stress for compression, S_{ws} the working unit stress for shear, t the thickness of the plates, d the diameter of rivet, p the pitch of rivets in a row, or the distance from centre to centre in a row, and n the number of rivets.

Diameter of Rivets.—Then the area of a rivet is $\frac{\pi d^2}{4} = 0.7854d^2$. The shearing resistance of a rivet is $0.7854d^2S_{ws}$, and the total shearing resistance of n rivets is $0.7854nd^2S_{ws}$. The bearing surface of a rivet is dt, of n rivets ndt, and the resistance to crushing $ndtS_{wc}$. For equal strength of crushing and shearing we have for *single shear*, or lap joint,

$$0.7854nd^2S_{ws} = ndtS_{wc}, \quad \text{or} \quad d = \frac{tS_{wc}}{0.7854S_{ws}}. \quad . \quad . \quad . \quad (1)$$

For double shear, or butt joint with two cover-plates, we have

$$1.5708nd^2S_{ws} = ndtS_{wc}, \quad \text{or} \quad d = \frac{tS_{wc}}{1.5708S_{ws}}. \quad . \quad . \quad . \quad (2)$$

For threefold shear we have 3×0.7854 in place of 0.7854 in (1), and so on.

It is customary to take $S_{wc} = 12500$ lbs. per square inch and $S_{ws} = 7500$ lbs. per square inch for wrought-iron rivets in single shear.

We have then

$$\left.\begin{array}{l} d = 2.12t \text{ for single shear;} \\ d = 1.06t \text{ for double shear.} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Practical Value of d.—Owing to risk of injury to the material in punching, the diameter of rivet *must always be at least as large as*

the thickness of the thickest plate through which it passes, and the diameter as given by (1), (2) or (3) must be chosen with reference to this restriction. The least allowable thickness of a plate is ¼ inch. We should have then as a lower limit for double shear, $d = \frac{1}{4}$ inch. But rivets as small as this are rarely used. Usually ½ inch is the least diameter allowable. A common practical rule is

$$d = 1\tfrac{1}{4}t + \frac{3}{16}, \quad \dots \dots \dots \quad (4)$$

where d is the diameter of rivet, and t the thickness of the plate in inches. When this rule gives d greater than (1), (2) or (3), we use it; otherwise we use (1), (2) or (3), *unless considerations of pitch*, as given in what follows, prevent.

Pitch of Rivets.—The area of plate between two rivets is $(p-d)t$; and if S_w is the working unit stress of tension or compression for the plates, and S_{ws} the working unit stress for shear, we have for equal strength:

for single shear or lap joint

$$(p-d)tS_w = \frac{\pi d^2}{4}S_{ws}, \quad \text{or} \quad p = d\left(1 + 0.7854\frac{dS_{ws}}{tS_w}\right);$$

for double shear or butt joint

$$(p-d)tS_w = \frac{\pi d^2}{2}S_{ws}, \quad \text{or} \quad p = d\left(1 + 1.5708\frac{dS_{ws}}{tS_w}\right).$$

Since S_{ws} and S_w are nearly equal, we have practically, if A is the area of cross-section of a rivet,

for single shear

$$p = d\left(1 + 0.7854\frac{d}{t}\right) = d + \frac{A}{t};$$

for double shear

$$p = d\left(1 + 1.5708\frac{d}{t}\right) = d + \frac{2A}{t}. \quad \dots \dots \quad (5)$$

The plate section is reduced by punching from pt between two rivets to $(p-d)t$, so that in the case of a tension joint the strength is reduced in the ratio

$$\frac{p-d}{p} = \frac{1}{1+\dfrac{4t}{\pi d}} \quad \text{or} \quad \frac{1}{1+\dfrac{2t}{\pi d}}.$$

We see at once that for a given thickness t a large rivet gives a large pitch and less reduction in strength than a small rivet. Small rivets allow a less pitch at a sacrifice of strength. But the less the pitch the tighter the joint. When strength rather than tightness is desired, as in bridges and parts of buildings and machines, we should then use a large rivet. When tightness is essential, as in steam-boilers, we should use a small rivet with a sacrifice of strength.

Practical Restrictions.—Owing to risk of injury to the material in punching and liability to tear out, *rivets are not allowed a pitch of less than 3 diameters*, or, if this distance is less then 3 inches, as it usually is, *less than 3 inches. Rivets should not be spaced farther apart than 6 inches* in any case, or, when the plate is in compression, 16 *times the thickness of the thinnest outside plate.* This last is to guard against buckling of the outside plate between rivets. With these restrictions we may apply (5).

Number of Rivets.—Guided by the preceding restrictions and

rules, we can select in any case a suitable size of rivet. This done, we can easily determine the number required.

A rivet is considered as failing either by shearing across or by crushing. In any case, then, the diameter being chosen, we must take such a number as shall give security against these two methods of failure, choosing the greater number. In general the number to resist crushing will be more than enough to resist shear. Still we should try for both. The bearing area of a rivet is the projection of the hole upon the diameter, or dt.

The allowable compressive stress is about 12500 lbs. per square inch. The allowable shear is taken at 7500 lbs. per square inch for single shear.

In the following Table we have given the safe shearing and bearing resistance for rivets of different sizes and for different thicknesses of plate. Having chosen, then, the size of rivet, an inspection of the Table will give its resistance. The stress to be resisted being known, the number to resist this stress either by bearing or shearing is easily determined. The greatest of these two numbers is taken, with enough over in any case to complete the row or rows. As most practical cases are double shear, the greatest number will usually be determined by the bearing resistance.

Distance from End to Edge.—The distance between the end and edge of any plate and the centre of rivet-hole, or between rows, is fixed by practice *at never less than* $1\frac{1}{4}$ *inches*, and when practicable it should be at least 2 diameters for rivets over $\frac{5}{8}$ inch diameter.

Joints in Compression.—The size and number of rivets are determined for joints in compression precisely as for joints in tension, because the joints are not considered as in contact and hence the rivets must transmit the stress in either case.

RIVET TABLE.

SHEARING AND BEARING RESISTANCE OF RIVETS.

Diameter of Rivet in inches.		Area of Rivet in square inches.	Single Shear at 7500 lbs. per square inch.	Bearing Resistance in pounds for Different Thicknesses of Plate at 12500 lbs. per square inch = $12500 \times dt$.										
Fraction.	Decimal.			$\frac{1}{4}''$	$\frac{5}{16}''$	$\frac{3}{8}''$	$\frac{7}{16}''$	$\frac{1}{2}''$	$\frac{9}{16}''$	$\frac{5}{8}''$	$\frac{11}{16}''$	$\frac{3}{4}''$	$\frac{13}{16}''$	$\frac{7}{8}''$
$\frac{3}{8}$	0.375	0.1104	828	1170	1465	1760								
$\frac{7}{16}$	0.4375	0.1503	1130	1370	1710	2050	2390							
$\frac{1}{2}$	0.5	0.1963	1470	1560	1950	2340	2730	3125						
$\frac{9}{16}$	0.5625	0.2485	1860	1760	2200	2640	3080	3520	3955					
$\frac{5}{8}$	0.625	0.3068	2300	1950	2440	2930	3420	3900	4390	4880				
$\frac{11}{16}$	0.6875	0.3712	2780	2150	2680	3220	3760	4290	4830	5370	5908			
$\frac{3}{4}$	0.75	0.4418	3310	2340	2930	3520	4100	4690	5270	5860	6440	7030		
$\frac{13}{16}$	0.8125	0.5185	3890	2540	3170	3800	4440	5080	5710	6350	6980	7620	8250	
$\frac{7}{8}$	0.875	0.6013	4510	2730	3420	4100	4780	5470	6150	6840	7520	8200	8890	9570
$\frac{15}{16}$	0.9375	0.6903	5180	2930	3660	4390	5130	5860	6590	7320	8050	8790	9520	10250
1	1	0.7854	5890	3125	3900	4690	5470	6250	7030	7810	8590	9370	10160	10940
$1\frac{1}{16}$	1.0625	0.8866	6650	3320	4150	4980	5810	6640	7470	8300	9130	9960	10790	11620
$1\frac{1}{8}$	1.125	0.9940	7460	3520	4390	5270	6150	7030	7910	8790	9667	10550	11420	12300
$1\frac{3}{16}$	1.1875	1.1075	8310	3710	4640	5570	6490	7420	8350	9280	10200	11130	12060	12990

Investigation and Designing of Beams. — From page 284 we must have for safety, as regards shearing, at every point of a beam

$$S_{ws}A \overset{=}{>} - V_x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(I)}$$

where A is the area of vertical cross-section at any point, S_{ws} is the working unit stress for shear and V_x is the vertical shearing force at any point, or the algebraic sum of all the vertical external forces between any point *and the left end*.

From page 288 we have

$$\frac{S_f I}{v} = \mp M_x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II)}$$

where S_f is the unit stress within the elastic limit in the most remote fibre of any cross-section at a distance v from the neutral axis, I is the moment of inertia of that cross-section with reference to the neutral axis, M_x is the bending moment at that cross-section of all the external forces on either side between the cross-section and either end, the minus sign being taken for forces on the left and the plus sign for forces on the right, and $\frac{S_f I}{v}$ is the resisting moment at the cross-section of the fibres *belonging to the left-hand portion of the beam*. If then this comes out minus we have compression in the bottom fibres, and if it comes out plus we have tension in the bottom fibres.

We have also, from page 288,

$$\frac{S_r I}{v} = \mp M_r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III)}$$

where S_r is the coefficient of rupture, or the breaking unit stress in the most remote fibre at the dangerous section, and M_r is the bending moment at that section.

From (III), if S_r is known, we can find in any case the breaking weight. Average values of S_r are given in the Table page 290.

When experiments upon S_r are lacking we may use a mean value between the ultimate tensile and compressive strength for approximate calculations. If we divide the breaking weight by the factor of safety (page 291), we obtain the allowable or working load.

From (II) we can find the load for any value of S_f within the elastic limit S_e (page 290). If we put for S_f the working unit stress S_w (page 292), we also obtain the working load.

We can also find from (II) the shape for uniform strength. The following cases will make plain the application of these equations.

Case 1. Cantilever Beam—Load W at the Free End. — Let l be the length of the beam and x the distance from the free end of any cross-section through the point C of the neutral *plane* (page 286).

Then the bending moment at that point is

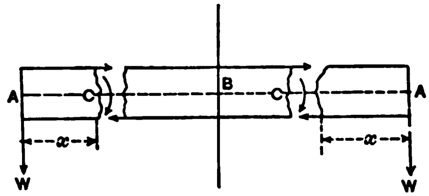
$$M_x = + Wx, \quad \text{or} \quad M_x = - Wx,$$

according as the weight W is on the left or right of the point P.

In both cases, then we have from (II), for the resisting moment of the fibres belonging to the *left-hand portion of the beam AC*,

$$\frac{S_f I}{v} = - Wx,$$

where I is the moment of inertia of the cross-section at C, and S_f is the stress in the most remote fibre of that cross-section at the distance v from the neutral axis. The minus sign denotes that we have compression in the *lower* fibre in both cases, as shown in the figure.



We have then, without reference to direction of rotation,

$$S_f = \frac{Wvx}{I}, \quad \ldots \ldots \ldots \ldots \quad (1)$$

or

$$W = \frac{S_f I}{vx}. \quad \ldots \ldots \ldots \ldots \quad (2)$$

From (2) we can find in any case the load W which will cause a given stress S_f in the most remote fibre of any cross-section at any distance x from the free end.

From (1) we can find the stress S_f for any given load W.

In any case we have only to substitute the value of v, x and I.

1. *Breaking Weight — Constant Cross-section.* — Rupture will occur at that section for which S_f is the greatest.

If I is constant, v is constant and we see from (1) that S_f will be greatest when $x = l$. The dangerous section is then at the fixed end. We have then from (III), page 298,

$$\frac{S_r I}{v} = - Wl,$$

where the minus sign denotes, as before, compression in the *lower* fibres and S_r is the coefficient of rupture. We have then, without reference to direction of rotation, for the breaking weight

$$W = \frac{S_r I}{vl}. \quad \ldots \ldots \ldots \ldots \quad (3)$$

If, for instance, the beam is rectangular in cross-section of breadth b and height h, then (page 278) $I = \frac{1}{12} bh^3$, $v = \frac{h}{2}$, and the breaking weight is

$$W = \frac{S_r bh^2}{6l}.$$

If the beam is triangular in cross-section of horizontal base b and height h, then (page 271) $I = \frac{bh^3}{36}$, $v = \frac{2}{3} h$, and the breaking weight is

$$W = \frac{S_r bh^2}{24l}.$$

In the same way we can find the breaking weight for any form of cross-section by substituting in (3) the value of I and v. The value of S_r can be taken from our Table page 290 for approximate determinations. We see that the strength of a beam is directly as the breadth and as the square of the height, and inversely as the length.

2. *Shape for Uniform Strength.*—Let the cross-section vary so that I is the moment of inertia of any cross-section at the distance x from the free end, and I_1 the moment of inertia of the cross-section at the fixed end.

Then from (1) the unit stress in the most remote fibre of any cross-section is

$$S_f = \frac{Wvx}{I},$$

where, v is the distance of that fibre from the neutral axis.

For the most remote fibre of the end cross-section we have then $S_f = \frac{Wv_1 l}{I_1}$, where v_1 is the distance of that fibre from the neutral axis.

Now for uniform strength *the outer fibre stress must be the same at every cross-section.* We have then for the condition of uniform strength

$$\frac{Wvx}{I} = \frac{Wv_1 l}{I_1}, \quad \text{or} \quad \frac{vx}{I} = \frac{v_1 l}{I_1}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If, for instance, the beam is rectangular in cross-section at every point, the breadth and height at the fixed end b_1 and h_1, and at any point b and h, we have (page 278)

$$I = \frac{1}{12}bh^3, \quad v = \frac{h}{2}; \qquad I_1 = \frac{1}{12}b_1h_1^3, \quad v_1 = \frac{h_1}{2};$$

and hence, from (4), we have for the condition of uniform strength

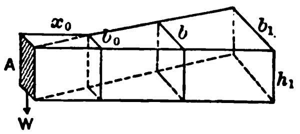
$$\frac{x}{bh^2} = \frac{l}{b_1h_1^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now if the height is constant, $h = h_1$, and we have for the breadth at any point distant x from the free end

$$b = \frac{b_1}{l}x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The breadth then varies as the ordinate to a straight line from b_1 at the fixed end to zero, theoretically, at the free end. Practically the breadth cannot be zero at the free end, but must have a value b_0 such that the area $A = b_0h_1$ at the free end may resist the shear.

We have then from (I), page 284, b_0h_1 at least equal to $\frac{W}{S_{ws}}$, or we must have b_0 at least equal to



$$b_0 = \frac{W}{h_1S_{ws}}.$$

Substituting this value of b_0 for b in (6), we find that the cross-section must be constant for a distance x_0 from the free end at least equal to

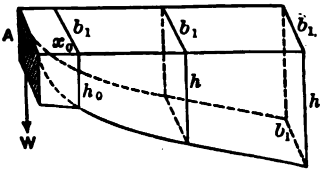
$$x_0 = \frac{Wl}{h_1b_1S_{ws}}.$$

For any value of x greater than x_0 the breadth is given by equation (6).

If the breadth is constant, $b = b_1$, and we have from (5), for the height at any point distant x from the free end,

$$h^2 = \frac{h_1^2}{l}x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The height then varies as the ordinate to a parabola from b_1 at the fixed end to zero, theoretically, at the free end. Here, again, we must have the height at the free end practically at least equal to



$$h_0 = \frac{W}{b_1 S_{ws}}.$$

Substituting this for h in (7), we find that the cross-section must be constant for a distance x_0 from the free end at least equal to

$$x_0 = \frac{W^2 l}{h_1^2 b_1^2 S^2_{ws}}.$$

For any value of x greater than x_0 the height is given by equation (7).

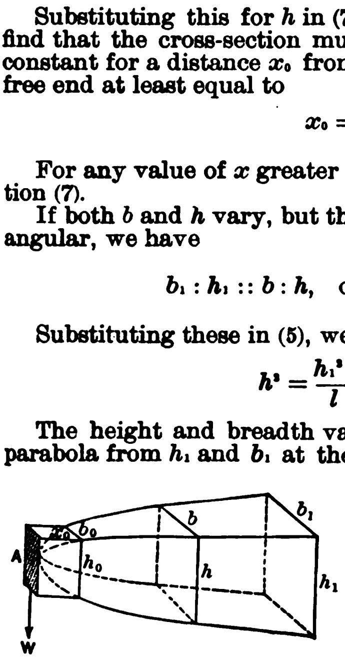
If both b and h vary, but the cross-section at every point is rectangular, we have

$$b_1 : h_1 :: b : h, \quad \text{or} \quad b = \frac{b_1 h}{h_1}, \quad h = \frac{h_1 b}{b_1}.$$

Substituting these in (5), we have

$$h^3 = \frac{h_1^3}{l}x, \quad b^3 = \frac{b_1^3}{l}x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The height and breadth vary then as the ordinates to a cubic parabola from h_1 and b_1 at the fixed end to zero, theoretically, at the free end. The area at any point is then, from (8),



$$bh = h_1 b_1 \sqrt[3]{\frac{x^2}{l^2}}.$$

The area A at the free end should be at least, from (I), page 284,

$$A = b_0 h_0 = \frac{W}{S_{ws}}.$$

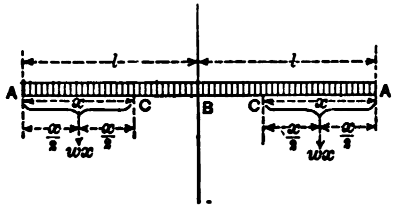
The cross-section should therefore be constant and equal to $b_0 h_0 = \frac{W}{S_{ws}}$ at least, for a distance x_0 from the free end given by

$$x_0 = \frac{Wl}{h_1 b_1 S_{ws}} \sqrt{\frac{W}{h_1 b_1 S_{ws}}}$$

For any value of x greater than x_0 the height and breadth are given by (8). Inserting the value of x_0 in (8), we obtain h_0 and b_0 at the free end.

In a similar way we can find the shape for uniform strength for any other form of cross-section, by substituting in (4) the values of I, I_1, v and v_1.

Case 2. Cantilever Beam—Load per Unit of Length w Uniformly Distributed.—The total load on the whole beam is $W = wl$. The load over any distance x from the free end is wx, and we can take it acting at its centre of mass or at $\frac{1}{2}x$ from the free end.



We have then for the bending moment at any point distant x from the free end

$$M_x = +\frac{wx^2}{2}, \quad \text{or} \quad M_x = -\frac{wx^2}{2},$$

according as the load wx is on the left or right of the point.

In both cases, then, we have from (II), for the resisting moment of the fibres belonging to the *left-hand portion of the beam AC*,

$$\frac{S_f I}{v} = -\frac{wx^2}{2}.$$

The minus sign denotes that we have compression in the *lower* fibres.

We have then, without reference to direction of rotation,

$$S_f = \frac{wvx^2}{2I}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$wx = \frac{2S_f I}{vx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (2) we can find the load which will cause a given stress S_f in the most remote fibre of any cross-section at a distance x from the free end. From (1) we can find the stress S_f for any given load wx. In any case we have only to substitute the value of I, x and d.

1. *Breaking Weight—Constant Cross-section.*—Rupture will occur at that section for which S_f is greatest. If I is constant, d is constant, and we see from (1) that S_f will be greatest when $x = l$. The dangerous section is then at the fixed end. We have then from (III), page 298,

$$\frac{S_r I}{v} = -\frac{wl^2}{2},$$

where the minus sign denotes, as before, compression in the *lower* fibres, and S_r is the coefficient of rupture. We have then, without reference to direction of rotation, for the breaking weight

$$W = wl = \frac{2S_r I}{vl}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or *twice as much as for the same beam with the same load W at the free end* (page 299).

If, for instance, the beam is rectangular in cross-section, of breadth b and height h, then (page 278) $I = \frac{1}{12}bh^3$, $v = \frac{h}{2}$, and the breaking weight is

$$W = wl = \frac{S_r bh^2}{3l},$$

If the beam is triangular in cross-section, of horizontal base b and height h, then (page 273) $I = \frac{bh^3}{36}$, $v = \frac{2}{3}h$, and the breaking weight is

$$W = wl = \frac{S_f bh^2}{12l}.$$

In the same way we can find the breaking weight for any form of cross-section by substituting in (3) the values of I and v.

2. *Shape for Uniform Strength.*—Let I be the moment of inertia at any cross-section and I_1 the moment of inertia at the fixed end, the distance of the outer fibre being v and v_1. Then for uniform strength we must have S_f at the end equal to S_f at any cross-section, or, from (1),

$$\frac{wvx^2}{2I} = \frac{wv_1 l^2}{2I_1}, \quad \text{or} \quad \frac{vx^2}{I} = \frac{v_1 l^2}{I_1}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For rectangular cross-section

$$I = \frac{1}{12}bh^3, \quad v = \frac{h}{2}; \qquad I_1 = \frac{1}{12}b_1 h_1^3, \quad v_1 = \frac{h_1}{2};$$

and hence

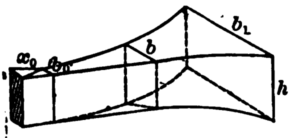
$$\frac{x^2}{bh^2} = \frac{l^2}{b_1 h_1^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

For constant height $h = h_1$ and

$$b = \frac{b_1}{l^2}x^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The breadth then varies as the ordinate to a parabola. From equation (I), page 284, we must have for the breadth b_0 at the distance x_0 from the free end

$$b_0 = \frac{wx_0}{h_1 S_{ws}}.$$



Substituting this in (6), we find that the cross-section must be constant for the distance x_0 from the free end at least equal to

$$x_0 = \frac{wl^2}{b_1 h_1 S_{ws}}.$$

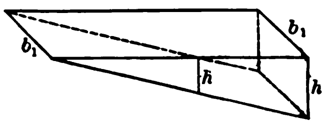
and the breadth at the free end is then

$$b_0 = \frac{w^2 l^2}{b_1 h_1^2 S_{ws}^2}.$$

For any value of x greater than x_0 the breadth is given by (6).

For constant breadth $b = b_1$ in (5) and

$$h = \frac{h_1}{l}x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$



From equation (I), page 284, we have for the height h_0 at the distance x_0 from the free end

$$h_0 = \frac{wx_0}{b_1 S_{ws}}.$$

Substituting this in (7), we find

that in order to resist shear we must have the end cross-section $A_1 = b_1h_1$ at least equal to

$$A_1 = b_1h_1 = \frac{wl}{S_{ws}}.$$

If then the end cross-section is safe for shear, every cross-section is safe, and for any value of x the height is given by (7).

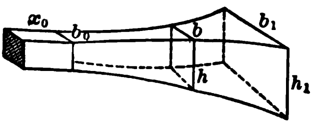
The height varies then as the ordinate to a straight line, from h_1 at the fixed end to zero at the free end.

If both b and h vary, we have for rectangular cross-section at every point

$$b_1 : h_1 :: b : h, \quad \text{or} \quad b = \frac{b_1h}{h_1}, \qquad h = \frac{h_1b}{b_1}.$$

Substituting in (5), we have for the height and breadth at any point

$$h^3 = \frac{h_1^3}{l^2}x^2, \quad b^3 = \frac{b_1^3}{l^2}x^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$



From equation (I), page 284, we must have at least

$$b_0^2h_0^2 = \frac{w^2x_0^2}{S^2_{ws}}.$$

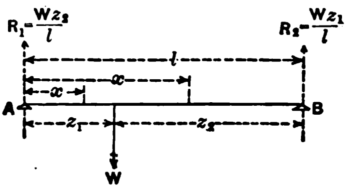
Hence, from (8), the cross-section must be constant and equal to $b_0h_0 = \frac{wx_0}{S_{ws}}$ at least, for a distance x_0 from the free end given by

$$x_0 = \frac{w^3l^4}{b_1^3h_1^3S^3_{ws}}.$$

For any value of x greater than x_0 the height and breadth are given by (8). Inserting the value of x_0 in (8), we obtain h_0 and b_0 at the free end.

In a similar way we can find the shape for uniform strength for any other form of cross-section by substituting in (4) the values of I, I_1, v and v_1.

Case 3. Beam Loaded with W Between the Supports.—Let l be the length of the beam, z_1 the distance of W from the left end and z_2 from the right end.



Then the left reaction $R_1 = \frac{Wz_2}{l}$.

For any point distant x from the left end we have for the bending moment (page 285),

$$\text{when } x < z_1, \quad M_x = -\frac{Wz_2}{l}x = -\frac{W(l - z_1)x}{l};$$

$$\text{when } x > z_1, \quad M_x = -\frac{Wz_2}{l}x + W(x - z_1) = -\frac{Wz_1(l - x)}{l}.$$

In each case, then, we have for the resisting moment, from (II), page 288,

$$\text{when } x < z_1, \quad \frac{S_fI}{v} = +\frac{W(l - z_1)x}{l}, \quad \text{or} \quad W = \frac{S_fIl}{vx(l - z_1)}; \quad (1)$$

$$\text{when } x > z_1, \quad \frac{S_fI}{v} = +\frac{Wz_1(l - x)}{l}, \quad \text{or} \quad W = \frac{S_fIl}{vz_1(l - x)}. \quad (2)$$

The plus sign denotes tension in the lower fibres.

From (1) and (2) we can find in any case the load W which placed at any given point will cause a given stress S_f in the most remote fibre of any cross-section at a distance x from the left end, or we can find the stress S_f for any given W. In any case we have only to substitute the value of I, v and x.

1. *Breaking Weight—Constant Cross-section.*—We see from (1) and (2) that for constant I and v, S_f is greatest when $x < z_1$ for the greatest value of x or $x = z_1$, and when $x > z_1$ for the least value of x or $x = z_1$. The dangerous section is then at the weight. We have then from (III), page 288,

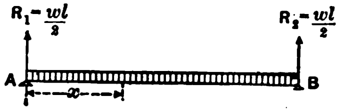
$$\frac{S_rI}{v} = \frac{Wz_1z_2}{l}, \quad \text{or} \quad W = \frac{S_rIl}{vz_1z_2}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or the same as for a cantilever beam of length z_1 with a load $\frac{Wz_2}{l}$ at the free end (page 299).

All the results of page 299 hold, then, in this case if we put $l = z_1$ and $W = \frac{Wz_2}{l}$. For the load at the middle of the beam $W = \frac{4S_rI}{vl}$, or four times as great as for a cantilever beam of the same length similarly loaded.

2. *Shape for Uniform Strength*—.The shape for uniform strength, in any case, is for each portion of the beam z_1 and z_2, precisely the same as for a cantilever beam of length z_1 or z_2 with the weight $\frac{Wz_2}{l}$ or $\frac{Wz_1}{l}$ at the free end, instead of W (page 300).

Case 4. Beam Loaded with w Uniformly Distributed.—The reaction at each end is $\frac{wl}{2}$.



For any point distant x from the left end the bending moment is

$$M_x = -\frac{wl}{2}x + \frac{wx^2}{2} = -\frac{wx}{2}(l - x).$$

The resisting moment is from (II), page 288, for the fibres belonging to the left-hand portion of the beam,

$$\frac{S_fI}{v} = +\frac{wx}{2}(l - x).$$

The plus sign denotes tension in the lower fibres.

We have then

$$S_f = \frac{wvx(l - x)}{2I}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$wx = \frac{2S_fI}{v(l - x)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

1. *Breaking Weight—Constant Cross-section.*—We see from (1) that for constant I and d, S_f is greatest when $x = (l - x)$ or $x = \frac{l}{2}$. The dangerous section is then at the middle of the span. We have then from (III), page 288,

$$\frac{S_rI}{v} = \frac{wl^2}{8}, \quad \text{or} \quad W = wl = \frac{8S_rI}{vl}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or eight times as much as for a cantilever beam with the same load W at the free end (page 299).

2. *Shape for Uniform Strength.*—Let I be the moment of inertia at any cross-section distant x from the left end, and I_1 at the middle of span, the distances of the outer fibre being v and v_1. Then, from (1), for uniform strength

$$\frac{wvx(l-x)}{2I} = \frac{wv_1l^2}{8I_1}, \quad \text{or} \quad \frac{vx(l-x)}{I} = \frac{v_1l^2}{4I_1}. \quad . \quad . \quad . \quad (4)$$

For rectangular cross-section

$$I = \frac{1}{12}bh^3, \quad v = \frac{h}{2}; \qquad I_1 = \frac{1}{12}b_1h_1^3, \quad v_1 = \frac{h_1}{2};$$

and hence

$$\frac{x(l-x)}{bh^2} = \frac{l^2}{4b_1h_1^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

For constant height $h = h_1$ and

$$b = \frac{4b_1}{l^2}x(l-x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The breadth then varies as the ordinate to a parabola, as on page 303, and the end cross-section must have a constant breadth

$$b_0 = \frac{wl}{h_1S_{ws}}\left(1 - \frac{wl}{4b_1h_1S_{ws}}\right)$$

for a distance from the left end

$$x_0 = l\left(1 - \frac{wl}{4b_1h_1S_{ws}}\right).$$

For any value of x greater than x_0 the breadth is given by (6).

In the same way we can find the shape for uniform strength when the breadth is constant, or when both b and h vary and the cross-section is rectangular, as on page 304. Or, by substituting in (4) the values of I, I_1, v and v_1, we can find the shape for uniform strength for any form of cross-section.

Theory of Pins and Eyebars.—The bearing resistance of a pin should equal the greatest pressure upon it due to any plate through which it passes.

Bearing.—If d is the diameter of pin, t the thickness of any plate through which it passes, then dt is the bearing area. Let S_{wc} be the working unit stress for compression, then dtS_{wc} is the bearing resistance of the pin. This should equal the stress transmitted by the plate, or

$$dtS_{wc} = \text{stress}.$$

We may take S_{wc} at 6.25 tons. The stress transmitted is always known. For a transmitted stress of *one ton* the required bearing area is then

$$dt = \frac{1}{6.25},$$

and hence we have

$$\textit{lineal bearing on pin per ton of stress} = \frac{1}{6.25d}. \quad . \quad . \quad (1)$$

From (1), having given the diameter d, we can find the corresponding lineal bearing or thickness of plate for every ton of transmitted stress. We have only to multiply this by the number of tons transmitted stress in any case to find the requisite thickness of the plate.

Diameter of Pin.—Let t be the thickness of plate or eyebar, and h its depth, then th is the area of cross-section of plate or eyebar. If S_{wt} is the working unit stress for tension, then thS_{wt} is the transmitted stress. Now if d is the diameter of the pin, and the thickness of the eyebar *head* is equal to the thickness of the bar, we have td for the bearing area of pin, and tdS_{wc} for its bearing resistance. We must have, then, for equal strength

$$tdS_{wc} = thS_{wt}, \quad \text{or} \quad d = \frac{S_{wt}}{S_{wc}}h.$$

We can take the ratio $\frac{S_{wt}}{S_{wc}} = \frac{3}{4}$. Hence the *least diameter of pin* is

$$d = \frac{3}{4}h. \qquad (2)$$

The diameter of pin may need to be greater than this, but it cannot be less, unless the thickness of eyebar *head* is made greater than the thickness of the bar itself.

When this is the case, if t_1 is the thickness of the bar and t the thickness of the head, we have for the least diameter of pin

$$tdS_{wc} = t_1hS_{wt}, \quad \text{or} \quad d = \frac{3}{4}\frac{t_1}{t}h, \qquad (3)$$

and for the thickness of head

$$t = \frac{3ht_1}{4d}. \qquad (4)$$

The pin is a round beam subjected to flexure. The size of pin as thus determined is greater than the diameter required for safe bearing or shearing. For a beam we have (page 288)

$$\frac{S_fI}{r} = M_{\text{max}},$$

where r is the radius of the pin and S_f is the unit stress in the outer fibre, and $I = \frac{\pi r^4}{4}$. Hence

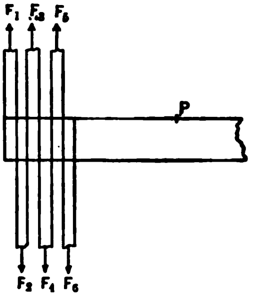
$$M_{\text{max}} = \frac{\pi S_f d^3}{32}, \qquad (5)$$

where M_{max} is the maximum bending moment. The usual value for S_f is 15000 lbs. per square inch for iron and 20000 lbs. per square inch for steel.

We have then, in any case, to find the maximum bending moment M_x, and then, from (5), we can find d.

Maximum Bending Moment.—In general for any pin, we must resolve the stress in every bar through which the pin passes into its vertical and horizontal components. The stress in each bar is considered as acting along the centre line or axis, and hence the point of application of each vertical and horizontal component is at the centre of the bearing of the corresponding bar.

Let M_h be the maximum bending moment of all the horizontal and M_v of all the vertical forces. Then the resultant maximum bending moment is



$$M_{max} = \sqrt{M_h^2 + M_v^2}.$$

From (5) we then find the diameter d of the pin.

Let the parallel horizontal or vertical components on one side of the centre of pin be F_1, F_2, F_3, F_4, etc., the odd indices F_1, F_3, etc., acting in one direction, and the even indices F_2, F_4, etc., acting in the other. Let l_1 be the distance between centres of bearing F_1 and F_2, l_2 the distance between F_2 and F_3, etc. We can now easily find the maximum moment by trial.

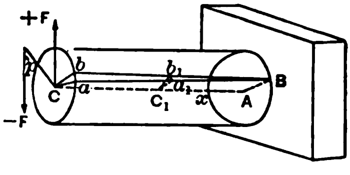
Thus the moment at F_2 is F_1l_1. Add to this $(F_1 - F_2)l_2$ and we have the moment at F_3. Add again $(F_1 - F_2 + F_3)l_3$ and we have the moment at F_4, and so on. The greatest of all these is the moment required.

Since all the forces F_1, F_3, F_5, etc., on one side are equal to all on the other, F_2, F_4, F_6, etc., they reduce to a couple on each side of centre of the pin, and hence the moment at any point P beyond the last force, as F_6, is constant. We have then only to find the greatest moment M_h or M_v by trial as directed.

Practical Sizes for Pins.—Pins are furnished in sizes differing by $\frac{1}{8}$ inch, and all sizes are an even number of sixteenths. A pin must always be ordered at least one sixteenth larger than the hole it is to fit, in order that it may be turned down to fit. We must then add $\frac{1}{16}$ inch to the calculated size, and if this gives an even number of sixteenths it can be ordered; if not, add $\frac{1}{16}$ more.

Thus if the size of a pin is $4\frac{3}{8}$ inches by calculation, it should be ordered at least $4\frac{7}{16}$; but since only even sixteenths are furnished, we should order $4\frac{1}{2}$ and turn down to fit the hole.

Torsion.—Torsion occurs when the external forces acting upon a body tend to twist it, so that each section turns on the next adjacent section about a common axis at right angles to the plane of section.



Let a horizontal shaft of length l be fixed at one end, and let a force couple $+F$, $-F$ act at the free end whose moment about the axis AC is Fp.

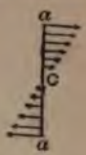
The shaft will be twisted about the axis AC so that any radial line as aC moves to bC through the angle $aCb = \theta$.

If the elastic limit is not exceeded, any longitudinal plane $aBAC$ before twisting remains plane after, as $bBAC$, and when the couple $+F$, $-F$ is removed the line bC returns to its original position aC. Also the angle aCb is proportional to F and to the distance $AC = l$ of the cross-section from the fixed end. Thus if θ is the angle aCb at the distance l from the fixed end, the angle $a_1C_1b_1$ at the distance x from the fixed end is $\frac{x}{l}\theta$. If the elastic limit is exceeded, this proportionality does not hold, the line bC does not return to its original position when the couple $+F$, $-F$ is removed, and if the twist is great enough we have rupture.

These facts are but a restatement of the general experimental laws of page 279.

Neutral Axis.—Consider the shaft to be made up of an indefinitely great number of parallel fibres. Since within the elastic limit stress is proportional to strain, as one cross-section of the shaft turns about the axis and slides upon the adjacent cross-section, the strain and therefore the shearing stress on each fibre of a cross-section is *proportional to its distance from the axis AC*. For the fibre at the axis AC there is then no shearing stress. The axis AC is then the **neutral axis.** (Compare page 286.)

Position of the Neutral Axis.—Let a be the cross-section of any fibre, and S_s the unit shearing stress within the elastic limit for that fibre in any cross-section *most remote from the neutral axis* at the distance v. Then the shearing stress for the most remote fibre in any cross-section at the distance v is $S_s a$, and for any other fibre in that cross-section, at the distance r, it is $\frac{r}{v}S_s a$. The sum of all the fibre stresses of any section in any straight line perpendicular to the axis is then $\frac{S_s}{v}\Sigma ra$.



But the sum of the external forces $+F$, $-F$ is zero, hence for equilibrium we must have $\Sigma ar = 0$.

Therefore the neutral axis AC *must pass through the centre of mass of the cross-sections.* (Compare page 287.)

Twisting Moment and Resisting Moment.—All the external forces acting upon the shaft reduce to a couple $+F$, $-F$, as shown in the figure, whose moment Fp with reference to the neutral axis is the **twisting moment** M_t. This moment is the same at every point of the neutral axis AC, and therefore tends to make each cross-section turn on its adjacent cross-section nearest the fixed end, about the axis AC, so that there must be for equilibrium between every two cross-sections an equal and opposite **resisting moment** due to the shearing stress between these two cross-sections.

Since for any cross-section the shearing stress for any fibre at a distance r from the neutral axis is $\frac{r}{v}S_s a$, the moment of that stress about the neutral axis is $\frac{S_s}{v}ar^2$, and the sum of the moments of all the stresses for any cross-section about the axis, or the resisting moment, is then $\frac{S_s}{v}\Sigma ar^2$.

For equilibrium this is balanced by the twisting moment M_t.

But Σar^2 is the *polar moment of inertia I_z of the cross-section* with reference to the axis through the centre of mass (page 271).

We have then for equilibrium, without reference to direction of rotation,

$$\frac{S_s I_z}{v} = M_t, \quad \ldots \ldots \ldots \ldots \quad (I)$$

where S_s is the unit shearing stress within the limit of elasticity in the most remote fibre of any cross-section at the distance v from the neutral axis, I_z is the polar moment of inertia of the cross-section with reference to that axis, which always passes through its centre of mass, and M_t is the twisting moment.

The student should note the analogy of this equation with that for flexure of beams, page 288.

From (I) we can find M_t for any given S_s when I_z and v are known and the elastic limit is not exceeded.

Coefficient of Rupture.—Equation (I) holds within the elastic limit. The value of S_s computed by means of (I) from experiments carried to the *point of rupture* we call the

Coefficient of Rupture for Torsion.—It is found by experiment to agree closely with the ultimate shearing strength as given in our Table page 290.

We have then for rupture

$$\frac{S_r I_z}{v} = M_t, \qquad \text{(II)}$$

where S_r is the shearing unit stress in the most remote fibre of that cross-section where rupture occurs, or the dangerous cross-section. This is evidently the cross-section for which $\frac{I_z}{v}$ is a minimum, since M_t is the same for every cross-section.

From (II) we can find M_t for S_r, I_z and v given, at the point of rupture.

Coefficient of Elasticity for Shearing Determined by Torsion.—Let the length of shaft be l and let the angle of torsion or the angle of twist of the end cross-section be θ and the twisting moment M_t. Then within the limit of elasticity the strain of the outer fibre for the end cross-section is $d\theta$ and the strain per unit of length is $s = \frac{d\theta}{l}$. The unit shearing stress of the outer fibre of the end cross-section is S_s. Then from page 281, since the coefficient of elasticity is the ratio of the unit stress to the unit strain,

$$E = \frac{S_s}{s} = \frac{lS_s}{v\theta},$$

where v is the distance of the outer fibre of the end cross-section from the neutral axis.

If we substitute for S_s its value from (I), we have

$$E = \frac{lM_t}{\theta I_z}, \qquad \text{(III)}$$

from which E can be computed if the other quantities are known and the elastic limit is not exceeded.

Inversely we have

$$\frac{E\theta I_z}{l} = M_t. \qquad \text{(IV)}$$

From (IV) we can find M_t for any given θ, when E, I_z and l are given and the elastic limit is not exceeded.

Work of Torsion.—If θ is the angle of torsion for any cross-section, the strain of any fibre in that cross-section at a distance r from the neutral axis is $r\theta$, and the stress for that fibre is $\frac{r}{v} S_s a$. The work of the fibre is then one half the product of the stress and strain (page 281), or $\frac{S_s\theta}{2v} ar^2$. The work of all the fibres is then $\frac{\theta S_s}{2v} \Sigma ar^2$; or, since $\Sigma ar^2 = I_z$, we have from (IV) and (I), for the work,

$$W = \frac{\theta S_s I_z}{2v} = \frac{M_t\theta}{2} = \frac{EI_z\theta^2}{2l} = \frac{M_t^2 l}{2EI_z}. \qquad \text{(V)}$$

Transmission of Power by Shafts.—Work is the product of a force by the distance through which it acts. Power is rate of work. A horse-power is 33000 ft.-lbs. of work per minute. If a shaft makes n revolutions per minute and the twisting force is F with a lever-arm p, then $2\pi p \times n$ is the distance and $2\pi npF$ is the work per minute, and the horse-power is, if p is in inches,

$$H = \frac{2\pi nFp}{33000 \times 12}.$$

But $Fp = M_t = \dfrac{S_s I_z}{v}$. Hence

$$H = \frac{\pi n S_s I_z}{198000v}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (VI)$$

where n is the number of revolutions per minute, H the horse-power transmitted, I_z and v must be taken in inches and S_s in pounds per square inch.

Combined Stresses.—We have thus far considered stresses of pure tension, compression and shear, also flexure and torsion. But we may have tension or compression combined with flexure, as when a beam is in direct longitudinal tension or compression and at the same time supports a load. We may also have tension or compression combined with shear, as when a shaft is in direct longitudinal compression or tension and at the same time in torsion. We may also have torsion and flexure combined.

Combined Tension and Flexure.—For flexure alone we have, page 288,

$$S_f = \frac{M_x v}{I},$$

where S_f is the unit stress in the extreme outer fibre in any cross-section at the distance v from the neutral axis. If this cross-section is also in direct tension, then the tensile fibre stresses due to flexure will be increased and the compressive fibre stresses due to flexure will be diminished. The neutral axis is then no longer at the centre of mass of the cross-section; and if we consider the deflection, a strict discussion leads to results of great complexity.

If, however, we neglect the deflection, and let T be the direct tension over the area A, then $\dfrac{T}{A}$ is the unit stress of direct tension. In the extreme outer tensile fibre, then, the total unit stress is $S_f + \dfrac{T}{A}$.

If S_{max} is the maximum unit stress, we have then at the cross-section where $S_f + \dfrac{T}{A}$ *is a maximum*

$$S_{max} = S_f + \frac{T}{A} = \frac{M_x v}{I} + \frac{T}{A}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where M_x is the bending moment at that cross-section of area A *for which* $S_f + \dfrac{T}{A}$ *is a maximum*, T is the direct tension, S_f is the unit stress due to flexure in the extreme outer tensile fibre of that cross-section at the distance v from the neutral axis.

From (1) we have

$$M_x = \frac{\left(S_{max} - \dfrac{T}{A}\right)I}{v}.$$

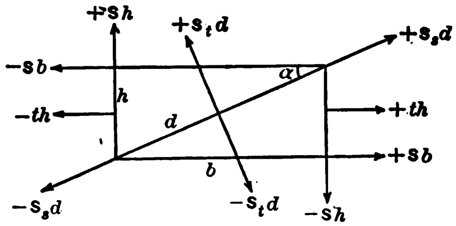
If we put for I its value $A\kappa^2$, where κ is the radius of gyration of the cross-section of area A, *for which* $S_f + \frac{T}{A}$ *is a maximum*, we have, putting S_{max} = the working unit stress S_w,

$$A = \frac{M_x v}{S_w \kappa^2} + \frac{T}{S_w}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) we can find in any case the maximum unit stress in the extreme outer fibre on the tensile side. From (2) we can find the area of cross-section by taking for S_w its value as found on page 291, by dividing the ultimate strength by the factor of safety, or as found by the method of page 292.

Combined Compression and Flexure.—This case is the same as the preceding, except that we must put the direct compression C in place of T and take for S_w the working stress for compression. If flexure is to be apprehended, we must take S_w as given on page 291.

Combined Tension and Shear.—If a body whose cross-section at any point is A is subjected to a direct tension T, the direct unit tensile stress is $t = \frac{T}{A}$. Suppose at the same time a direct vertical shear S, then the unit shearing stress is $s = \frac{S}{A}$.



Take any element of breadth b, height h and unit thickness. Then we have acting on this element the tensile stresses $+ th$, $- th$, and the shearing stresses $+ sh$, $- sh$. The two equal and opposite stresses $+ th$, $- th$ hold each other in equilibrium. The couple $+ sh$, $- sh$ can only be held in equilibrium by the opposite couple $+ sb$, $- sb$. Let d be the diagonal, and α the angle of the diagonal with the side b. Then we have the components parallel to the diagonal forming the combined shearing stresses $+ s_s d$, $- s_s d$, and the components perpendicular to the diagonal forming the combined tensile stresses $+ s_t d$, $- s_t d$.

For equilibrium we have then

$$+ s_s d - th \cos \alpha - sb \cos \alpha + sh \sin \alpha = 0;$$
$$+ s_t d - th \sin \alpha - sb \sin \alpha - sh \cos \alpha = 0.$$

Since we have $\sin \alpha = \frac{h}{d}$, $\cos \alpha = \frac{b}{d}$, dividing these equations by d, we obtain

$$s_s = t \sin \alpha \cos \alpha + s \cos^2 \alpha - s \sin^2 \alpha = \frac{t}{2} \sin 2\alpha + s \cos 2\alpha;$$

$$s_t = t \sin^2 \alpha + 2s \sin \alpha \cos \alpha = \frac{t}{2} - \frac{t}{2} \cos 2\alpha + s \sin 2\alpha.$$

From these equations, by placing the first differential coefficient equal to zero, we have, when s_s is a maximum,

$$\tan 2\alpha = \frac{t}{2s}, \quad \sin 2\alpha = \frac{t}{\sqrt{4s^2 + t^2}}, \quad \cos 2\alpha = \frac{2s}{\sqrt{4s^2 + t^2}};$$

when s_t is a maximum,

$$\tan 2\alpha = -\frac{2s}{t}, \quad \sin 2\alpha = -\frac{2s}{\sqrt{4s^2 + t^2}}, \quad \cos 2\alpha = \frac{t}{\sqrt{4s^2 + t^2}}.$$

Therefore we have

$$\max s_s = \sqrt{s^2 + \frac{t^2}{4}}; \quad \dots\dots\dots \quad (1)$$

$$\max s_t = \frac{t}{2} + \sqrt{s^2 + \frac{t^2}{4}}. \quad \dots\dots\dots \quad (2)$$

Equation (1) gives the unit shearing stress when we have the direct unit tensile stress t and unit shearing stress v combined. Equation (2) gives the unit tensile stress when we have the direct tensile stress t and unit shearing stress v combined.

Combined Compression and Shear.—Let the direct unit compressive stress be c, and the direct unit shearing stress be s. Then, just as before, we have for the combined unit shearing stress

$$s_s = \sqrt{s^2 + \frac{c^2}{4}}, \quad \dots\dots\dots \quad (1)$$

and for the combined unit compressive stress

$$s_c = \frac{c}{2} + \sqrt{s^2 + \frac{c^2}{4}}. \quad \dots\dots\dots \quad (2)$$

Combined Flexure and Torsion.—Let S_f be the greatest unit stress for flexure as given by equation (II), page 288, viz.,

$$S_f = \frac{M_x v}{I},$$

and S_s the unit shearing stress for torsion as given by equation (I), page 309, viz.,

$$S_s = \frac{M_t v}{I_z}.$$

Then, as we have just seen, we have for the combined unit stresses of shear and compression or tension

$$s_s = \sqrt{S_s^2 + \frac{S_f^2}{4}};$$

$$s_t \text{ or } s_c = \frac{S_f}{2} + \sqrt{S_s^2 + \frac{S_f^2}{4}}.$$

Stress Due to Temperature.—We have from equation (3), page 281,

$$\lambda = \frac{lS}{E},$$

where λ is the strain produced by the unit stress S in a bar of length l, the coefficient of elasticity being E.

If a bar is constrained so that it cannot change in length and then exposed to change of temperature, a unit stress will be produced equal to that which would cause a strain equal to the change of length of the unconstrained bar under the same change of temperature.

Thus if ϵ is the coefficient of linear expansion for one degree of temperature, t the number of degrees of change of temperature and l the original length, the change of length of an unconstrained bar is $\lambda = \epsilon t l$. The strain per unit of length is then $\frac{\lambda}{l} = \epsilon t$. The coefficient of linear expansion $\epsilon = \frac{\lambda}{lt}$ is then the ***strain per unit of length per degree.***

If the bar is constrained so that it cannot change its length, we then have a unit stress

$$S = \frac{E\lambda}{l} = E\epsilon t,$$

which is independent of the length l. The total stress, if the area is A, is then

$$AS = AE\epsilon t.$$

We give the following average values of the coefficient of linear expansion ϵ for one degree Fahrenheit:

Brick and stone.............	$\epsilon = 0.0000050$
Cast iron....................	$\epsilon = 0.0000062$
Wrought iron...............	$\epsilon = 0.0000067$
Steel.........................	$\epsilon = 0.0000065$

EXAMPLES.

(1) *A wrought-iron tie-rod,* 30 *ft. long and* 4 *sq. in. in area of cross-section, is subjected to* 40000 *lbs. tension. Find the unit stress. If the coefficient of elasticity is* 30000000 *lbs. per square inch, find the elongation.*

Ans. Unit stress = 10000 lbs. per square inch. Elongation = 0.01 ft.

(2) *An iron bar* 10 *ft. long has a strain of* 0.012 *ft. under a unit stress of* 25000 *lbs. per square inch. Find the coefficient of elasticity.*

Ans. $E = 20833333$ lbs. per square inch.

(3) *A rectangular timber tie is* 12 *inches deep and* 40 *ft. long. If* $E = 1200000$ *lbs. per square inch, find the thickness so that the elongation under a pull of* 270000 *lbs. may not exceed* 1.2 *inches.*

Ans. Thickness = 7.5 in.

(4) *A wrought-iron tie-rod* 142 *ft. long and* 4 *sq. in. area is subjected to a stress of* 80000 *lbs. If* $E = 30000000$ *lbs. per square inch, find the elongation.*

Ans. Elongation = 1.136 in.

(5) *The length of a cast-iron pillar is diminished from* 20 *ft. to* 19.97 *ft. under a given load. Find the unit stress of compression, E being* 17000000 *lbs. per square inch.*

Ans. Unit stress = 25500 lbs. per square inch.

(6) *A wrought-iron bar 2 sq. in. area of cross-section has its ends confined between two immovable blocks at a temperature of 60° Fahr. Taking the coefficient of expansion at* 0.000006944, *find the pressure upon the blocks when the temperature is* 100° *Fahr., supposing there is no flexure.*

Ans. Pressure = 0.00055552 E. If E = 30000000 lbs. per square inch, pressure = 16665.6 lbs.

(7) *The dead load of a bridge is* 5 *tons and the live load* 10 *tons per panel, the corresponding factors of safety being* 3 *and* 6. *Find the combined factor of safety.*

Ans. Factor = 5.

(8) *The dead load upon a short hollow cast-iron pillar, with a rectangular area of* 20 *sq. in., is* 50 *tons. If the compression is not to exceed* 0.0015 *of the length, find the greatest live load, E being* 17000000 *lbs. per square inch.*

Ans. Live load = 410000 lbs. = 205 tons.

(9) *A steel suspension rod in a suspension bridge carries* 3500 *lbs. of roadway and* 3000 *lbs. of live load. Its length is* 30 *ft. and sectional area one half square inch. Find the gross load and the extension of the rod, E being* 35000000 *lbs. per square inch.*

Ans. Gross load = 6500 lbs. Extension 0.133 inch.

(10) *A beam* 40 *ft. long carries a load of* 20000 *lbs. Find the shearing force at* 15 *ft. from one end, and also the maximum bending moment:* (*a*) *when the beam is supported at the ends and loaded in the middle;* (*b*) *when it is supported at the ends and loaded uniformly;* (*c*) *when it is fixed at one end and loaded at the other;* (*d*) *when it is fixed at one end and loaded uniformly.*

Ans. (*a*) Shear = 10000 lbs., max. moment = 200000 ft.-lbs. at middle;
(*b*) Shear = 2500 lbs., max. moment = 100000 ft.-lbs. at middle;
(*c*) Shear = 20000 lbs., max. moment = 800000 ft.-lbs. at end;
(*d*) Shear = 7500 lbs., max. moment = 400000 ft.-lbs. at end.

Draw the diagrams for shear and bending moment in each case.

(11) *A beam* 20 *ft. long rests on two supports and carries a load of* 10 *tons at* 5 *ft. from one end. Find the maximum bending moment.*

Ans. Maximum moment 37.5 ft.-tons at the weight. Draw the diagrams for shear and bending moment.

(12) *Find the breadth and depth of the strongest rectangular beam which can be cut from a cylindrical log of diameter D*

Ans. Breadth $= D\sqrt{\frac{1}{3}}$, depth $= D\sqrt{\frac{2}{3}}$.

(13) *A round and a square beam are equal in length and equally loaded. Find the ratio of the diameter to the side of the square, so that the two beams may be of equal strength.*

Ans. $\dfrac{\text{Diameter}}{\text{Side}} = 2\sqrt[3]{\dfrac{2}{3\pi}}$.

(14) *Compare the relative strengths of a cylindrical beam and the strongest rectangular and square beams that can be cut from it.*

Ans. $\dfrac{\text{Strength of cylindrical}}{\text{Strongest rectangular}} = \dfrac{9\pi\sqrt{3}}{32} = 1.53;$

$\dfrac{\text{Strength of cylindrical}}{\text{Strongest square}} = \dfrac{3\pi\sqrt{2}}{8} = 1.66.$

(15) *Compare the relative strengths of a solid square beam to that of the solid inscribed cylinder.*

Ans. $\dfrac{\text{Strength of square}}{\text{Strength of cylinder}} = \dfrac{16}{3\pi} = 1.7.$

(16) *Compare the strength of a square beam with its sides vertical to that of the same beam with a diagonal vertical.*

Ans. $\dfrac{\text{Side vertical}}{\text{Diagonal vertical}} = \sqrt{2} = 1.414.$

(17) *A beam of yellow pine, 14 inches wide, 15 inches deep, resting upon supports 10 ft. 9 in. apart, was just able to bear a weight of 34 tons at the centre. What weight at the centre will a beam of the same material, 3 ft. 9 in. between the supports and 5 inches square bear?*

Ans. 3.86 tons.

(18) *Compare the strengths of two rectangular beams of equal length, the breadth and depth of one being respectively equal to the depth and breadth of the other.*

Ans. The strengths are directly as the breadths and inversely as the depths.

(19) *A cast-iron beam 4 inches square rests upon supports 6 ft. apart. Find the breaking weight at the centre, taking $S_r = 30000$ lbs. per square inch.*

Ans. Breaking weight = $17777\frac{7}{9}$ lbs.

(20) *A yellow-pine beam, 14 inches wide, 15 inches deep, resting upon supports 10 ft. 6 in. apart, broke down under a uniformly-distributed load of 60.97 tons. Find the coefficient of rupture S_r.*

Ans. $S_r = 3658.2$ lbs. per square inch.

(21) *A cast-iron rectangular beam rests upon supports 12 ft. apart and carries a weight of 2000 lbs. at the centre. If the breadth is one half the depth, find the sectional area so that the unit stress may nowhere exceed 4000 lbs. per square inch.*

Ans. Area = 18 sq. in., depth = 6 inches, breadth = 3 inches.

(22) *A wrought-iron beam, 4 inches deep, $\frac{3}{4}$ inch wide, fixed horizontally at one end, gave way when loaded with 1568 lbs. at the free end, at a point 2 ft. 8 in. from the load. Find the coefficient of rupture S_r.*

Ans. $S_r = 25088$ lbs. per square inch.

(23) *A wrought-iron beam 2 inches wide and 4 inches deep rests upon supports 12 ft. apart. Find the uniformly distributed load it will carry in addition to its own weight if $S_r = 50000$ lbs. per square inch and the factor of safety is 4. A bar of iron 3 ft. long and one square inch in cross-section weighs 10 lbs.*

Ans. Load = 3384 lbs.

(24) *Find the length of a beam of ash 6 inches square which would break of its own weight when supported at the ends, the weight of the timber being 30 lbs. per cubic foot and $S_r = 7000$ lbs. per square inch.*

Ans. Length = $149\frac{2}{3}$ ft.

(25) *A cast-iron cantilever beam 8 ft. long and 12 inches deep, centre to centre of the flanges, carries a uniformly-distributed load*

[illegible] lbs. *Find the area of the top flange at the fixed end, neglecting the web, so that the unit stress shall not exceed [illegible] lbs. per square inch.*

Ans. Area = [illegible] square inches.

(26) *A cast-iron beam [illegible] inches deep, centre to centre of the flanges, rests upon supports [illegible] ft. apart. Its bottom flange is [illegible] inches wide and [illegible] inches deep. Neglecting the web, find the breaking weight at the centre, the coefficient of rupture S_r being [illegible] lbs. per square inch.*

Ans. Weight = [illegible] lbs.

(27) *A cantilever plate girder [illegible] ft. long and [illegible] ft. deep, centre to centre of the flanges, supports a uniform load of 1.[illegible] tons per foot and a weight of [illegible] tons at the free end. Find the unit stress on the net section of the tension flange at the point of support, neglecting the web, the gross area being [illegible] inches but reduced by rivet-holes two [illegible].*

Ans. Unit stress = [illegible] tons per square inch.

(28) *A girder [illegible] ft. long and 4 ft. deep, centre to centre of flanges, supports a uniform load of [illegible] tons. Find the stress in either flange at [illegible] feet from one end, neglecting the web.*

Ans. Stress = [illegible] tons.

(29) *Required the depth of a rectangular beam supported at the ends and carrying a load W at the middle, in order that the elongation of the lowest fibre shall equal [illegible] of its original length.*

Ans. Depth = $\sqrt{\frac{[illegible]}{E[illegible]}}$.

(30) *A beam of depth 8 inches, length 8 ft., supported at ends, sustains 500 lbs. per foot. Find its breadth for a factor of safety of 10, S_r being 14000 lbs. per square inch.*

Ans. Breadth = $3\frac{1}{7}$ inches.

(31) *A beam of length 12 ft., breadth 2 in., depth 5 in., is supported at the ends. Find the uniform load it will safely sustain for a factor of safety of 4, S_r being 8000 lbs. per square inch.*

Ans. Weight = 9259 lbs.

(32) *A wooden beam of length 12 ft. is supported at the ends. Find its breadth and depth so that it may safely sustain one ton uniformly distributed over its whole length, for the factor of safety 10, S_r being 15000 lbs. per square inch and the depth 4 times the breadth.*

Ans. Breadth = 2.08 in.; depth = 8.32 in.

(33) *A wrought-iron beam 12 ft. long, 2 in. wide, 4 in. deep is supported at the ends. The material weighs $\frac{1}{4}$ lb. per cubic inch. Taking S_r at 54000 lbs., find the uniform load it will sustain.*

Ans. Without the weight of beam, 16000 lbs.
Over the weight of beam, 15712 lbs.

(34) *A beam is fixed horizontally at one end. Length 20 ft., breadth $1\frac{1}{2}$ in., $S_r = 40000$ lbs. per square inch. If the weight of the material is $\frac{1}{4}$ lb. per cubic inch, find the depth so that it may just sustain its own weight and 500 lbs. at the free end.*

Ans. Depth = 4.05 inches.

(35) *Find the sectional area of a square beam of 12 ft. span which sustains a load of 300 lbs. at the centre and has at the same time a direct longitudinal tension of 2000 lbs.; the working unit stress being taken at 1000 lbs. per square inch.*

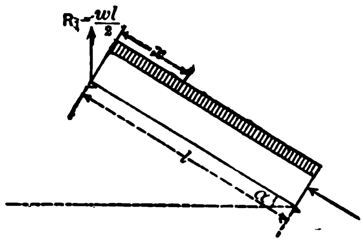
Ans. 4.18 inches square.

(36) *Find the sectional area of a square beam of 12 ft. span which sustains a load of 50 lbs. per foot uniformly distributed and has at the same time a direct longitudinal tension of 2000 lbs.; the working unit stress being taken at 1000 lbs. per square inch.*

Ans. 4.18 inches square.

(37) *A beam of uniform cross-section A is inclined at the angle α to the horizontal and rests without slipping on two supports. The load is w per linear unit, uniformly distributed. Find the maximum unit stress.*

Ans. This is the case of a roof-truss rafter at the bottom or at an intermediate panel, loaded by its own weight only.



The vertical reaction at the top end is given by

$$R_1 \cos\alpha \times l = wl \times \frac{1}{2} l \cos\alpha,$$

or

$$R_1 = \frac{wl}{2}.$$

The bending moment at any point distant x from the upper end is then

$$M_x = \frac{wl}{2}\cos\alpha \times x - wx \times \frac{x\cos\alpha}{2}.$$

The unit stress in the outer fibre at the distance v from the neutral axis is then for any cross-section at a distance x from the upper end

$$S_f = \frac{M_x v}{I} = \frac{dw\cos\alpha}{2I}(lx - x^2).$$

The direct compression at the distance x from the upper end is

$$C = wx\sin\alpha - \frac{wl}{2}\sin\alpha = \frac{w\sin\alpha}{2}(2x - l).$$

The combined unit stress is then

$$S_f + \frac{C}{A} = \frac{vw\cos\alpha}{2I}(lx - x^2) + \frac{w\sin\alpha}{2A}(2x - l).$$

This is a maximum when $x = \frac{l}{2} + \frac{I\tan\alpha}{Av}$.

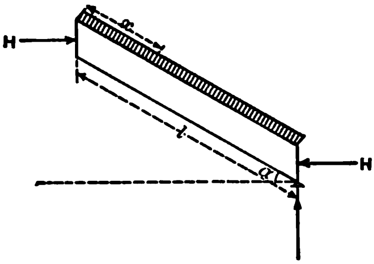
Hence the maximum unit stress is

$$S_{\max} = \frac{vwl^2\cos\alpha}{8I} + \frac{Iw\tan\alpha\sin\alpha}{2A^2v}.$$

If there is an additional compression applied at the ends of C, the maximum unit stress is $\frac{C}{A} + S_{\max}$.

(38) *The top rafter of a roof-truss of uniform cross-section A is inclined at the angle α to the horizontal. The load is w per linear unit uniformly distributed. Find the maximum unit stress.*

Ans. The reaction at the top end H is horizontal. We have then



$$Hl \sin \alpha = wl \times \frac{l \cos \alpha}{2},$$

or

$$H = \frac{wl}{2} \cot \alpha.$$

At any point x from the upper end the unit stress for flexure is then

$$S_f = \frac{(Hx \sin \alpha - \frac{wx^2}{2} \cos \alpha)v}{I}.$$

The direct compression is $C = H \cos \alpha + wx \sin \alpha$.
The combined unit stress is then

$$S_f + \frac{C}{A} = \frac{vw \cos \alpha}{2I}(lx - x^2) + \frac{wl \cot \alpha \cos \alpha}{2A} + \frac{wx \sin \alpha}{A}.$$

This is a maximum when $x = \frac{l}{2} + \frac{I \tan \alpha}{Av}$.
Hence the maximum unit stress is

$$S_{max} = \frac{vwl^2 \cos \alpha}{8I} + \frac{wl \operatorname{cosec} \alpha}{2A} + \frac{Iw \tan \alpha \sin \alpha}{2A^2 v}.$$

If there is an additional compression applied at the ends of C, the maximum unit stress is $\frac{C}{A} + S_{max}$.

(39) *A wooden beam* 10 *inches wide,* 9 *inches deep and* 8 *ft. long carries a uniform load of* 500 *lbs. per linear foot and is subjected to a longitudinal compression of* 40000 *lbs. Find the maximum unit stress.*

Ans. 800 lbs. per square inch.

(40) *If the beam in Example* (39) *forms one of the panels of the rafter of a roof-truss of* 40 *ft. span and* 15 *ft. high, find the maximum unit stress.*

Ans. Let b = breadth, h = height of cross-section.

Then $v = \frac{h}{2}$, $A = bh$, $I = \frac{1}{12}bh^3$, and we have, from Example (37),

$$\text{maximum unit stress} = \frac{40000}{bh} + \frac{3wl^2 \cos \alpha}{4bh^2} + \frac{w \tan \alpha \sin \alpha}{12b}.$$

In the present case $w = \frac{500}{12}$, $l = 96$, $b = 10$, $h = 9$, $\sin \alpha = 0.6$, $\cos \alpha = 0.8$, $\tan \alpha = 0.75$. Hence

maximum unit stress = 729 lbs. per square inch.

(41) *A rivet* ¾ *inch in diameter is subjected to a tension of* 2000 *lbs. and at the same time to a shear of* 3000 *lbs. Find the combined maximum tensile and shearing unit stresses and the angles they make with the axis of the rivet.*

Ans. Maximum shearing unit stress = 7155 pounds per square inch, making an angle of 9° 13′ with the axis of the rivet.
Maximum tensile unit stress = 9420 pounds per square inch, making an angle of 54° 28′ with the axis of the rivet.

(42) *A circular shaft 2 ft. long is twisted through an angle of 7 degrees by a couple of ± 200 lbs. with a lever-arm of 6 inches. Find the angle for a shaft of the same size and material 4 ft. long when twisted by a couple of 500 lbs. with a lever-arm of 18 inches.*

Ans. 105 degrees.

(43) *A circular shaft when twisted by a couple of ± 90 lbs. with a lever-arm of 27 inches has a unit shearing stress of 2000 lbs. per square inch. If the same shaft is twisted by a couple of ± 40 lbs. with a lever-arm of 57 inches, what is the unit shearing stress?*

Ans. 1877 pounds per square inch.

(44) *An iron shaft 5 ft. long and 2 inches diameter is twisted through an angle of 7 degrees by a couple of ± 5000 lbs. with a lever-arm of 6 inches, and on the removal of the couple springs back to its original position. Find the value of E for shearing.*

Ans. 9390000 pounds per square inch.

(45) *What is the couple which acting with a lever-arm of 12 inches will twist asunder a steel shaft 1.4 inches diameter, the coefficient of rupture by torsion being 75000 lbs. per square inch.*

Ans. ± 1683 pounds.

(46) *Compare the strength of a square shaft with that of a circular shaft of equal area.*

Ans. $\dfrac{\sqrt{2\pi}}{3}$.

(47) *Find the combined unit stresses for a wrought-iron shaft 3 inches diameter and 12 feet long, resting on bearings at each end, which transmits 40 horse-power while making 120 revolutions per minute, upon which a load of 800 pounds is brought by a belt and pulley at the middle.*

Ans. The unit stress for flexure is

$$S_f = \frac{M_x d}{I} = \frac{Wl}{\pi r^3} = 10800 \text{ lbs. per square inch.}$$

The unit stress for torsion is

$$S_s = \frac{198000 dH}{n\pi I_z} = 4000 \text{ lbs. per square inch.}$$

The maximum combined unit stresses are then :

for tension or compression, $5400 + \sqrt{4000^2 + 5400^2} = 12100$ lbs. per square inch; for shear 6700 lbs. per square inch.

(48) *A vertical shaft weighing with its loads 6000 lbs. is subjected to a twisting moment by a force of 300 pounds acting with a lever-arm of 4 feet. If the shaft is of wrought iron 4 feet long and 2 inches in diameter, find its maximum unit stress, provided the shaft is so supported that it cannot bend sideways.*

Ans. Compressive unit stress = 10170 lbs. per square inch.
Shearing " " = 9215 " " " "

(49) *Find the diameter of a short vertical steel shaft to carry a load of 6000 lbs. when twisted by a force of 300 lbs. with a leverage of 4 ft., taking unit stress for shear at 7000 lbs. and for compression at 10000 lbs. per square inch.*

Ans. About 2.5 inches.

(50) *A cast-iron water-pipe 12 inches diameter and $\frac{5}{8}$ in. thick is under a head of 300 ft. Taking the ultimate strength at 20000 lbs. per square inch, find the factor of safety.*

Ans. The unit pressure is $0.434 \times 300 = 130.2$ lbs. per square inch. Hence the unit stress is $S = \dfrac{130.2 \times 12}{2 \times \frac{5}{8}} = 1230$ lbs. per square inch. The factor of safety is then $\dfrac{20000}{1230} =$ about 16.

(51) *Find the thickness of a cast-iron pipe 18 inches diameter for a factor of safety of 10, taking the ultimate strength at 20000 lbs. per square inch and the head of water 300 feet.*

Ans. 0.586 inch.

(52) *A wrought-iron pipe, 4.5 inches internal diameter, weighs 12.5 pounds per linear foot. What pressure can it carry with a factor of safety of 8, taking the ultimate strength 55000 lbs. per square inch?*

Ans. A bar of wrought iron one square inch in cross-section and 3 ft. long weighs 10 lbs. Hence the area of the pipe metal is $12.5 \times \dfrac{3}{10} = 3.75$ square inches. The thickness is then $t = \dfrac{3.75}{2\pi r} = \dfrac{1}{4}$ inch.

Hence $p = \dfrac{2 \times 55000t}{8d} = 763$ lbs. per square inch.

(53) *A boiler is to be made of wrought-iron plates $\frac{3}{8}$ inch thick, united by single lap-joints. Find the size and pitch of rivets. If the boiler is 30 inches in diameter and carries a pressure of 100 lbs. per square inch above the atmosphere, find the factor of safety, taking the ultimate strength at 55000 lbs. per square inch.*

Ans. From (4), page 296, we have $\frac{5}{8}$-in. rivets. But from (3), page 295, we have $\frac{3}{4}$-in. This size would be chosen for ordinary construction work. In this case we wish a tight joint, and therefore use a small rivet at sacrifice of strength. Let us take then $\frac{5}{8}$-in. rivets. Then from (5), page 296, we find the pitch $\frac{3}{4}$ in. But this violates the practical restriction that rivets should not have a less pitch than three diameters. We take the pitch then 2 inches. The pressure on a length equal to the pitch is $30 \times 2 \times 100 = 6000$ lbs. If S is the unit stress, the resisting stress is $S\left(2 - \dfrac{5}{8}\right)t = \dfrac{33}{64}S$. Hence $S = \dfrac{64 \times 6000}{33} = 11640$ lbs. per square inch. The factor of safety is then about 5. If this is considered too small, we should use a less pitch or a larger rivet. A larger rivet would not be tight enough. For a less pitch the holes must be drilled and not punched.

(54) *Required to unite two $\frac{1}{2}$-inch plates by a butt joint with two cover-plates; the stress to be transmitted being 40000 lbs. and the unit working stress 10000 lbs. per square inch.*

Ans. The area of the plates must then be 4 square inches *net* if the joint is in tension, *gross* if in compression. The cover-plates can be each $\frac{1}{4}$ inch thick. Our rule (4), page 296, gives for diameter of rivet $d = \frac{13}{16}$ inch. This is greater than given by (3), page 295, therefore we take it. From our Table page 297 we have for the resistance to shear of a $\frac{13}{16}$-inch rivet 3890 lbs. The rivets are in double shear in a butt joint, hence we require $\dfrac{20000}{3890} =$ about 5 rivets. The bearing resistance from our Table is 5080 lbs. We require then for bearing $\dfrac{40000}{5080} =$ about 8 rivets. This, then, is the number we should use.

For the pitch we have from (5), page 296, 2.887 inches. This is less than 3 inches. We therefore take the pitch 3 inches. We must have at least $1\frac{1}{4}$ inches for distance from end and edge (page 297).

If the plates are $8\frac{1}{4}$ inches wide, we must then have three rows of rivets, three in the first and last and two in the middle on each side of the joint. The cover-plates must then be 10 inches long. The student can now sketch the cover-plates with the rivet-holes properly spaced.

(55) *A plate girder is 17 feet long and 27 inches deep. The uniformly-distributed load is 55,000 lbs. The thickness of the web is $\frac{1}{4}$ inch and of the flange angles $\frac{3}{16}$ inch. Find the size, number and spacing of the rivets to unite the web and flanges.*

Ans. From (4), page 296, we have $d = \frac{1}{2}$ inch. This is less than the size given by (3), page 295. We take the rivets then $\frac{7}{8}$ inch diameter.

If we neglect the web, the stress of compression in the upper flange or of tension in the lower, at any point distant x feet from the end, is given by

$$\frac{55000x}{4.5}\left(1 - \frac{x}{17}\right).$$

If we take $x = 0$, 2.5 ft., 5 ft., 8.5 ft., we have the stress at these points $= 0$, 26062 lbs., 43137 lbs, 51944 lbs.

We have then for the first division of 2.5 ft. the horizontal stress 26062 lbs., or 13 tons, to be taken by the rivets.

In the second division of 2.5 ft. we have $43137 - 26062 = 17075$ lbs., or 8.5 tons; and in the third division of 8.5 ft. we have $51944 - 43137 = 8807$ lbs., or 4.4 tons, to be taken by the rivets.

For the shear at any point distant x feet from the end we have

$$\frac{55000}{2}\left(1 - \frac{2x}{17}\right).$$

If we take $x = 0$, 2.5 ft., 5 ft., 8.5 ft., we have the shear at these points, $=$ 27500 lbs., 19400 lbs., 11300 lbs., 0.

We have then for the first division of 2.5 ft. the shear $27500 - 19400 = 8100$ lbs., or 4 tons, to be taken by the rivets.

In the second division of 2.5 ft. we have $19400 - 11300 = 8100$ lbs., or 4 tons; and in the third division of 8.5 ft. we have 11300 lbs., or 5.65 tons, to be taken by the rivets.

Hence the *combined* shear (page 313) in the first division of 2.5 feet is

$$\sqrt{4^2 + \frac{13^2}{4}} = 7.63 \text{ tons} = 15260 \text{ lbs.}$$

In the second division of 2.5 ft.,

$$\sqrt{4^2 + \frac{8.5^2}{4}} = 5.9 \text{ tons} = 11800 \text{ lbs.}$$

In the third division of 8.5 ft.,

$$\sqrt{5.65^2 + \frac{4.4^2}{4}} = 6 \text{ tons} = 12000 \text{ lbs.}$$

The bearing resistance of a seven-eighths inch rivet is, from our Table page 297, 2730 lbs. We require then for bearing, in the first 2.5 feet, $\frac{15260}{2730} = 6$ rivets, in the next 2.5 ft., $\frac{11800}{2730} = 5$ rivets, in the third division of 8.5 ft., $\frac{12000}{2730} = 5$ rivets.

We must not pitch the rivets less than 3 inches or more than 6 inches (page

296). A pitch of 4 inches for the first 2.5 ft., then 5 inches for the next 2.5 ft. and then 6 inches to the middle will therefore give more rivets than are necessary.

(56) *A pin 3 inches diameter passes through the web of a channel bar three fifths of an inch thick. The transmitted stress is 55500 lbs. Find the thickness of re-enforcing plate necessary to give sufficient bearing ou the pin.*

Ans. The thickness for each ton (page 306 (*b*)) is

$$\frac{1}{6.25d} = \frac{1}{6.25 \times 3} = 0.0533 \text{ inch.}$$

For 55500 lbs. = 27.75 tons we should have a thickness of 0.0533 × 27.75 = 1.48 inches.

The channel web is only $\frac{3}{5} = 0.6$ inch thick. In order to have the proper thickness for safe bearing on the pin, we must then increase the thickness by 1.48 − 0.6 = 0.88 inch. Two re-enforcing plates on each side of the web, each 0.44 inch thick or about $\frac{7}{16}$ inch each, will then give the required thickness.

(57) *If the depth of an eyebar is 10 inches, find the least diameter of pin which can be used without having the thickness of the head greater than that of the bar.*

Ans. (Page 307 (*c*).) $d = 7\frac{1}{2}$ inches.

(58) *A bar 8 in. by $\frac{7}{8}$ in. has a pin $4\frac{5}{8}$ inches diameter passing through it. Find the thickness of bar head.*

Ans. The least diameter without having the head thicker than bar is 6 inches. As the pin is less than this, the head must be thicker than the bar and equal to

$$t = \frac{3ht_1}{4d} = \frac{3 \times 8 \times \frac{7}{8}}{4 \times 4\frac{5}{8}} = 1\tfrac{5}{37} \text{ inches.}$$

(59) *In a panel of a bridge truss we have at each end of the pin two eyebars on one side, 4 in. by $1\frac{3}{16}$ in., and on the other side one eyebar 4 in. by $1\frac{7}{16}$ in. Also one tie on each side of centre of pin $1\frac{3}{16}$ in. thick. The tie is packed close to the vertical post, which consists of two channels of $\frac{7}{8}$-in. thickness. The bars are packed snug. The vertical compression in the half post is 40000 lbs. The working unit stress of the bars is 10000 lbs. per square inch. Find the size of pin required.*

Ans. We have here on one side acting horizontally

$$F_1 = F_2 = 4 \times 1\tfrac{3}{16} \times 10000 = 47500 \text{ lbs.,}$$

and on the other side

$$F_3 = 4 \times 1\tfrac{7}{16} \times 10000 = 57500 \text{ lbs.}$$

The horizontal component of the tie-stress is

$$F_4 = 2 \times 47500 - 57500 = 57500 \text{ lbs.}$$

The distances are

$$l_1 = l_2 = \frac{1}{2}(1\tfrac{3}{16} + 1\tfrac{7}{16}) = 1\tfrac{5}{16} \text{ inches;}$$

$$l_3 = \frac{1}{2}(1\tfrac{3}{16} + 1\tfrac{3}{16}) + \frac{7}{8} = 2\tfrac{1}{8} \text{ inches.}$$

We have then at F_2 the moment $F_1 l_1 = 47500 \times 1\frac{5}{16} = 62344$ inch-lbs.;

at F_3 we have $62344 + (F_1 - F_2)l_2 = 49219$ inch-lbs.;

at F_4 we have $49219 + (F_1 - F_2 + F_3)l_3 = 133594$ inch-lbs.

The maximum horizontal bending moment is then

$$M_h = 133594 \text{ inch-lbs.} = 66.797 \text{ inch-tons.}$$

The vertical compression in post is 40000 lbs. Its lever-arm is

$$\frac{1}{2}\left(1\tfrac{9}{16} + \frac{7}{8}\right) = 1\tfrac{7}{32}.$$

Hence

$$M_v = 40000 \times 1\tfrac{7}{32} = 48750 \text{ inch-lbs.} = 24.375 \text{ inch-tons.}$$

The resultant maximum bending moment is then

$$M_{\max} = \sqrt{M_h^2 + M_v^2} = \sqrt{66.8^2 + 24.4^2} = 71.11 \text{ inch-tons} = 142220 \text{ inch-lbs.}$$

We have then for size of pin about $4\frac{5}{8}$ inches diameter, or $4\frac{3}{4}$ commercial size. The least allowable diameter is $\frac{3}{4}h = 3$ inches. Hence the bearing is abundant.

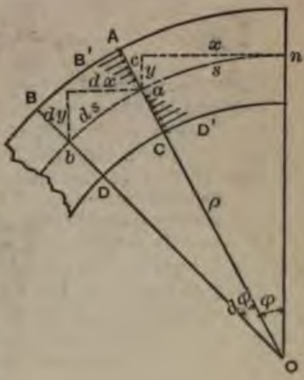
CHAPTER III.

APPLICATIONS OF STATICS—THEORY OF FLEXURE.

CHANGE OF SHAPE OF NEUTRAL AXIS OF A BEAM. ASSUMPTIONS OF THE THEORY. APPLICATION OF EQUATION I. DEFLECTION AND BREAKING WEIGHT OF BEAMS. DEFLECTION OF A FRAMED STRUCTURE. DEFLECTION OF BEAMS FOUND BY THE PRINCIPLE OF WORK. FORMULAS FOR LONG STRUTS.

Change of Shape of Neutral Axis of a Beam.—Let a beam be deflected from its original straight line by external forces, as shown in the figure.

Let the two sections AC and BD be consecutive plane sections parallel before flexure and remaining plane after.



Let the length of the neutral axis of the beam $na = s$, then the indefinitely small distance $ba = ds$. Let ϕ be the angle AOn. Then $d\phi$ is the angle BOA.

If the deflection is small, we can take $na = s$ equal to x, and $ab = ds$ equal to dx.

Let the bending moment at the point a of the neutral axis of the beam of the external forces be M_x, let S_f be the stress in the most remote fibre of any cross-section AC at the distanee v from the neutral axis of the cross-section at a, and I be the moment of inertia of the cross-section AC with reference to the neutral axis of the cross-section at a.

Then, as proved page 288 (a), the resisting moment of the fibre stresses at the cross-section AC is $\frac{S_f I}{v}$, and we have

$$\frac{S_f I}{v} = \mp M_x, \quad \ldots \ldots \ldots \ldots \quad (1)$$

where we take the minus sign if we take M_x for all external forces *on the left* of AC, and the plus sign if we take M_x for all external forces *on the right* of AC. If then M_x comes out minus, it indicates compression in the bottom fibres as in the figure; if plus, tension in the bottom fibres.

Now the strain in the most remote fibre at the distance v from the neutral axis, we see from the figure, is $v d\phi$, and the unit strain is then $\frac{v d\phi}{ds}$, or, since we can take dx for ds, $\frac{v d\phi}{dx}$. The unit stress in this fibre is

S_f. Since the coefficient of elasticity E is equal to the unit stress divided by the unit strain (page 281), we have

$$E = \frac{S_f}{\frac{v d\phi}{dx}}, \quad \text{or} \quad S_f = \frac{E v d\phi}{dx}.$$

Hence we have

$$\frac{EId\phi}{dx} = \mp M_x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But we see from the figure that $\frac{dy}{dx}$ equals the tangent of the angle ϕ. Since the deflection is very small, we can take the tangent as equal to the arc, and hence $\phi = \frac{dy}{dx}$. Therefore $d\phi = \frac{d^2y}{dx}$, and hence, from (2),

$$EI\frac{d^2y}{dx^2} = \mp M_x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From similar triangles we also have $v d\phi : v :: ds : \rho$, where ρ is the radius of curvature at a. Since we can take dx for ds, we have $\frac{d\phi}{dx} = \frac{1}{\rho}$. Hence, from (2),

$$\frac{EI}{\rho} = \mp M_x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

We have then

$$\frac{S_f I}{v} = \frac{EI}{\rho} = EI\frac{d^2y}{dx^2} = \mp M_x. \quad . \quad . \quad . \quad . \quad . \quad . \quad (I)$$

These are the fundamental equations of the theory of flexure.

The first of these equations, (1), we have already deduced in the preceding chapter, page 288, and have used it to find breaking weight and shape for uniform strength for ordinary cases of beams (page 299). From (4) we can find in any case the radius of curvature of the beam at any point. From (3) we can find the deflection at any point of a beam. Equation (3) is then the differential equation of the curve of deflection.

Thus by the application of one or the other of equations (I) all questions of flexure can be solved.

Assumptions of the Theory.—The assumptions upon which the theory of flexure as expressed by equations (I) rests should be clearly recognized. Thus we have assumed:

1st. That the deflection is very small, so that we can put x for s, dx for ds, $\frac{dy}{dx}$ for ϕ.

2d. That a section plane before flexure remains plane after flexure.

3d. That the elastic limit is not exceeded.

4th. That the coefficient of elasticity E is constant.

Upon these assumptions the theory rests. Comparison of its results with the results of experiment shows that within the elastic limit the theory is reliable.

Application of Equations (I).—The first of equations (I),

$$\frac{S_f I}{v} = \mp M_x,$$

we have already seen how to apply in the preceding chapter.

The second of equations (I),

$$\frac{EI}{\rho} = \mp M_x,$$

needs no special explanation.

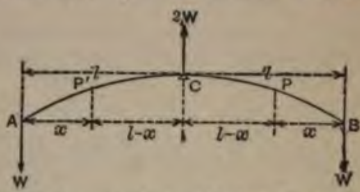
The third of equations (I),

$$EI\frac{d^2y}{dx^2} = \mp M_x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

requires a little general explanation before we proceed to its special applications.

In equation (1), $EI\frac{d^2y}{dx^2}$ is the *resisting moment* at any cross-section, that is, the algebraic sum of the moments of the fibre forces in the cross-section at any point with reference to the neutral axis of that cross-section. These fibre forces are always considered as belonging to that portion of the beam on the left of the cross-section. The *bending moment*, or the algebraic sum of the moments of all the external forces either on the right or left of the cross-section at any point, is denoted by M_x. We always consider a moment positive when it tends to cause counter clockwise rotation, and negative when it tends to cause clockwise rotation. In any case, then, we can write the algebraic sum denoted by M_x with the proper sign for each term, whether we take M_x for all forces on the left or on the right. We then use in (1) the minus sign when M_x is taken for all forces on *the left*, and the plus sign when M_x is taken for all forces on *the right*, of the cross-section at any point.

Thus, for example, take a beam AB of length $2l$, resting on the support C at its centre, with a load W at each end. The upward reaction is then $2W$. Let ACB represent the slightly deflected neutral axis of the beam.



For any point P' of the neutral axis of the beam distant x from the left end A we have, taking the algebraic sum of the moments of all external forces *on the left* of P',

$$M_x = +Wx,$$

where the plus sign indicates counter-clockwise rotation. If, however, we take the algebraic sum of the moments of all external forces *on the right* of P', we have $M_x = -W(l + l - x) + 2W(l - x) = -Wx$, where the minus sign denotes clockwise rotation. In the first case we use in (1) the minus sign, in the second case we use in (1) the plus sign. We therefore write *for both* cases, as we evidently ought to,

$$EI\frac{d^2y}{dx^2} = -Wx.$$

Again, take any point P distant x from the right end B. Here we have for the algebraic sum of the moments of all external forces on *the left* of P

$$M_x = W(l + l - x) - 2W(l - x) = +Wx,$$

and for the algebraic sum of the moments of all external forces on *the right* of P we have $M_x = -Wx$. In the first case we use in (1) the minus sign, in the second case we use in (1) the plus sign. We therefore again write for *both cases*

$$EI\frac{d^2y}{dx^2} = -Wx.$$

We obtain then in any given case the same expression from (1) for $EI\frac{d^2y}{dx^2}$, or the resisting moment of the fibre forces of the beam on the left of P, no matter where we take P, and no matter whether we take M_x for all forces on the left or right of P.

The minus sign for Wx in the present case denotes compression in the *lower* fibre. If the sign had come out plus, it would denote tension in the lower fibre, because in each case the sign gives the direction of rotation of the fibre moments of the beam on left of the section. This is in accord with the principle of the Differential Calculus that $\frac{d^2y}{dx^2}$ is minus or plus according as a curve is concave downwards or upwards. In the present case the curve of deflection is concave downwards.

The *vertical shearing force* at any section (page 288) is the algebraic sum of all the vertical forces *on the left* of that section. At any cross-section whose abscissa is x the bending moment is M_x and the vertical shear is V_x. At the next consecutive section the moment is

$$M_x + dM_x = M_x \mp V_x dx, \quad \text{or} \quad \frac{dM_x}{dx} = \mp V_x.$$

Hence from (1) we have

$$EI\frac{d^3y}{dx^3} = \frac{dM_x}{dx} = \mp V_x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where the minus sign is taken when dx is negative and the plus sign when dx is positive.

If we put $\frac{dM_x}{dx} = 0$, we obtain the value of x for which M_x is a maximum or a minimum. Hence the bending moment is either a maximum or a minimum at the point where the shear is zero.

If we integrate (1), we obtain

$$\frac{dy}{dx} = \mp \int_0^x \frac{M_x dx}{EI} + \text{Const.}$$

When $x = 0$, $\frac{dy}{dx}$ is the tangent t of the angle which the tangent to the curve at the origin makes with the axis of X. Hence Const. $= t$ and

$$\frac{dy}{dx} = t \mp \int_0^x \frac{M_x dx}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we put $\frac{d^2y}{dx^2} = 0$ or, from (1), $M_x = 0$, we obtain the value of x for which $\frac{dy}{dx}$ is a maximum or a minimum. Hence the tangent to the curve has either its maximum or a minimum inclination at the point where the bending moment M_x is zero.

If we integrate (3), we obtain

$$y = tx \mp \int_0^x dx \int_0^x \frac{M_x dx}{EI} + \text{Const.}$$

For $x = 0$, y is the deflection y_0 at the origin. Hence Const. $= y_0$ and

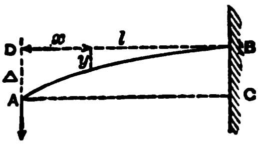
$$y = tx + y_0 \mp \int_0^x dx \int_0^x \frac{M_x dx}{EI}. \quad \ldots \ldots \quad (4)$$

If we put $\frac{dy}{dx} = 0$, we obtain the value of x for which y is a maximum or a minimum. Hence the deflection is either a maximum or a minimum at the point where the tangent to the curve is horizontal.

Let us now apply these principles to special cases.

Case 1. Cantilever Beam—Fixed Horizontally at One End—Load W at the Other End.—We have already seen how to find the breaking weight and shape for uniform strength in this case (page 299). It remains to find the deflection.

(a) Deflection—Uniform Cross-section.—Let the beam of length $AB = l$ be fixed horizontally at one end B and carry the load W at the other end A. Take the origin at the end D before deflection, and let x be the distance to any cross-section at P.



We have then for the bending moment at any point P of the neutral axis, taking moments on the left of P as in the figure, $M_x = + Wx$. Hence, from (I), page 326,

$$-M_x = EI\frac{d^2y}{dx^2} = -Wx. \quad \ldots \ldots \quad (1)$$

If the cross-section is constant, I is constant. We have then, by integrating (1),

$$EI\frac{dy}{dx} = -\frac{Wx^2}{2} + C_1. \quad \ldots \ldots \quad (2)$$

Integrating (2), we have

$$EIy = -\frac{Wx^3}{6} + C_1 x + C_2. \quad \ldots \ldots \quad (3)$$

The curve APB must pass through B, and the tangent at B must be horizontal. Hence we must have $y = 0$ for $x = l$ in (3) and $\frac{dy}{dx} = 0$ for $x = l$ in (2). If then we make $\frac{dy}{dx} = 0$ and $x = l$ in (2), we have $C_1 = +\frac{Wl^2}{2}$. If we make $y = 0$ and $x = l$ in (3), we have $C_2 = -\frac{Wl^3}{3}$. Substituting these values of the constants of integration in (2) and (3), we have

$$EI\frac{dy}{dx} = \frac{W}{2}(l + x)(l - x); \quad \ldots \ldots \quad (4)$$

$$EIy = -\frac{W}{6}(2l + x)(l - x)^2. \quad \ldots \ldots \quad (5)$$

Equation (4) gives the tangent of the angle which the tangent to the curve at any point makes with the horizontal. Equation (5) gives the deflection y for any point P of the neutral axis distant x from the free end. The maximum deflection Δ is evidently at the free end. Making, then, $x = 0$ in (5), we have for the maximum deflection

$$\Delta = -\frac{Wl^3}{3EI}. \quad \ldots \ldots \quad (6)$$

The minus sign shows that the deflection $\Delta = AD$ is downwards or below the horizontal through the origin D. From (6) we can find the deflection for any form of cross-section, according to the value of I. Thus for rectangular cross-section of breadth b and height h, $I = \frac{1}{12}bh^3$ (page 277) and

$$\Delta = -\frac{4Wl^3}{Ebh^3}.$$

[The student should solve this case taking the origin at B, C and A. He should also draw the figure with the load W at the right end and take the origin at A, B, C and D.]

(*b*) **Deflection—Beam of Uniform Strength.**—If the beam is of uniform strength, I is no longer constant. Suppose, for instance, a rectangular cross-section, the breadth and depth at the fixed end being b_1 and h_1. Then for constant height we have (page 300) for the breadth b at any point distant x from the free end $b = b_1\frac{x}{l}$. Hence $I = \frac{1}{12}b_1h_1^3\frac{x}{l}$, and from (1) we have

$$-M_x = \frac{d^2y}{dx^2} = -\frac{12Wl}{Eb_1h_1^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Integrating this we have

$$\frac{dy}{dx} = -\frac{12Wlx}{Eb_1h_1^3} + C_1; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$y = -\frac{6Wlx^2}{Eb_1h_1^3} + C_1x + C_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Making $\frac{dy}{dx} = 0$ for $x = l$ in (2), we have $C_1 = \frac{12Wl^2}{Eb_1h_1^3}$; and making $y = 0$ for $x = l$ in (3), we have $C_2 = -\frac{6Wl^3}{Eb_1h_1^3}$. Hence

$$\frac{dy}{dx} = \frac{12Wl}{Eb_1h_1^3}(l - x); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$y = -\frac{6Wl}{Eb_1h_1^3}(l - x)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The greatest deflection is at the free end and equal to

$$\Delta = -\frac{6Wl^3}{Eb_1h_1^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

or $\frac{3}{2}$ times as much as for beam of constant cross-section.

If we take the cross-section rectangular and the breadth constant, we have (page 301) for the height h at any point distant x from the free end $h = h_1\sqrt{\frac{x}{l}}$. Hence $I = \frac{1}{12}b_1h_1^3\sqrt{\frac{x^3}{l^3}}$, and

$$-M_x = \frac{d^2y}{dx^2} = -\frac{12Wl\sqrt{l}}{Eb_1h_1^3\sqrt{x}}.$$

Integrating twice and determining the constants of integration as before, we obtain

$$\frac{dy}{dx} = \frac{24Wl}{Eb_1h_1^3}\left(\sqrt{l} - \sqrt{lx}\right);$$

$$y = -\frac{8Wl}{Eb_1h_1^3}\left(l^2 - 3lx + 2x\sqrt{lx}\right).$$

For the greatest deflection at the free end we have

$$\Delta = -\frac{8Wl^3}{Eb_1h_1^3},$$

or twice as much as for beam of constant cross-section.

For similar rectangular cross-sections we have (page 301) $b = \sqrt[3]{\frac{b_1^3x}{l}}$, $h = \sqrt[3]{\frac{h_1^3x}{l}}$. Hence $I = \frac{1}{12}b_1h_1^3\frac{x}{l}\sqrt[3]{\frac{x}{l}}$, and

$$-M_x = \frac{d^2y}{dx^2} = -\frac{12Wl\sqrt[3]{l}}{Eb_1h_1^3\sqrt[3]{x}}.$$

Integrating twice and determining the constants of integration as before, we have

$$\frac{dy}{dx} = \frac{18Wl}{Eb_1h_1^3}\left(l - \sqrt[3]{lx^2}\right);$$

$$y = -\frac{18Wl}{5Eb_1h_1^3}\left(2l^2 - 5lx + 3\sqrt[3]{lx^5}\right).$$

For the greatest deflection at the free end we have

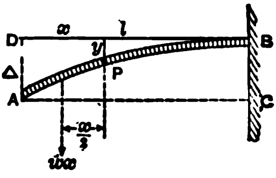
$$\Delta = -\frac{36Wl^3}{5Eb_1h_1^3},$$

or nine fifths as much as for beam of constant cross-section.

The volume of the beam in the first case is $\frac{2}{3}$, in the second case $\frac{1}{2}$ and in the third case $\frac{5}{9}$ of the volume of a rectangular beam of uniform cross-section. Hence the deflection at the end for a rectangular beam of uniform strength is proportional to the volume of the beam.

Case 2. Cantilever Beam—Fixed Horizontally at one End—Load Uniformly Distributed.—Here again we have already found the breaking weight and shape for uniform strength (page 302). It remains to find the deflection.

(a) Deflection—Uniform Cross-section.—Let w be the load per unit of length uniformly distributed, l the length AB of the beam, and take the origin at the end D before deflection.



Since we can take the load wx as acting at its centre of mass or at a distance $\frac{x}{2}$ from P, we have for the moment at P

$$M_x = wx \times \frac{x}{2} = \frac{wx^2}{2},$$

and from (I), page 326,

$$-M_x = EI\frac{d^2y}{dx^2} = -\frac{wx^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the cross-section is constant, I is constant. We have then by integrating (1)

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + C_1; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$EIy = -\frac{wx^4}{24} + C_1x + C_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The curve APB must pass through B, and the tangent at B must be horizontal. Hence we have $y = 0$ for $x = l$ in (2) and $\frac{dy}{dx} = 0$ for $x = l$ in (3). The constants are then $C_1 = +\frac{wl^3}{6} C_2 = -\frac{wl^4}{8}$ and

$$EI\frac{dy}{dx} = \frac{w}{6}(l^3 - x^3); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$EIy = -\frac{w}{24}(x^4 - 4l^3x + 3l^4). \quad . \quad . \quad . \quad . \quad (5)$$

The maximum deflection is at the free end and equal to

$$\varDelta = -\frac{wl^4}{8EI} = -\frac{Wl^3}{8EI}$$

if we put the load $wl = W$, or only $\frac{3}{8}$ as great as for an equal load at the end.

[The student should note this case, taking the origin at B, C, and A. He should also draw the figure with the fixed end on left and take the origin at A, B, C and D.]

(*b*) **Deflection—Beam of Uniform Strength.**—For uniform strength I is not constant. If we take the cross-section rectangular, the breadth and depth at the fixed end being b_1 and h_1, we have (page 303) for constant height for the breadth b at any point distant x from the free end $b = b_1\frac{x^2}{l^2}$. Hence $I = \frac{b_1h_1^3x^2}{12l^2}$ and

$$-M_x = \frac{d^2y}{dx^2} = \frac{6wl^2}{Eb_1h_1^3}$$

Integrating this twice and determining the constants of integration as before, we have

$$\frac{dy}{dx} = \frac{6wl^2}{Eb_1h_1^3}(l - x);$$

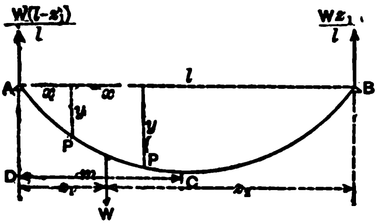
$$y = -\frac{3wl^2}{Eb_1h_1^3}(l - x)^2.$$

The deflection at the end is then

$$\varDelta = -\frac{3wl^4}{Eb_1h_1^3},$$

or 24 times as much as for the same beam of constant cross-section. In the same way we can find the deflection for breadth constant and for similar cross-sections.

Case 3. Horizontal Beam Loaded with W between the Supports—Constant Cross-section. — Let l be the length of the beam, z_1 the distance of W from the left end, and take the origin at the left end. (For breaking weight see page 305.)



The reaction at the left end is $\frac{W(l-z_1)}{l}$, and we have from (I), page 326, for any point P of the neutral axis distant x from the left end,

when $x < z_1$ $$M_x = EI\frac{d^2y}{dx^2} = +\frac{W(l-z_1)}{l}x = Wx - \frac{Wz_1x}{l}; \quad . \quad (1)$$

when $x > z_1$

$$-M_x = EI\frac{d^2y}{dx^2} = +\frac{W(l-z_1)}{l}x - W(x-z_1) = Wz_1 - \frac{Wz_1x}{l}. \quad . \quad (2)$$

If the cross-section is constant, I is constant.

Integrating (1), we obtain

$$\text{for } x < z_1 \quad EI\frac{dy}{dx} = \frac{Wx^2}{2} - \frac{Wz_1x^2}{2l} + C_1. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Integrating (2), we obtain

$$\text{for } x > z_1 \quad EI\frac{dy}{dx} = Wz_1x - \frac{Wz_1x^2}{2l} + C_2. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Integrating again, we obtain from (3)

$$\text{for } x < z_1 \quad EIy = \frac{Wx^3}{6} - \frac{Wz_1x^3}{6l} + C_1x + C_3, \quad . \quad . \quad . \quad (5)$$

and from (4)

$$\text{for } x > z_1 \quad EIy = \frac{Wz_1x^2}{2} - \frac{Wz_1x^3}{6l} + C_2x + C_4. \quad . \quad . \quad . \quad (6)$$

The curve APB must pass through A and B, and each portion AP and PB must have a common tangent and deflection at P. Hence we must have $y = 0$ for $x = 0$ in (5) and $x = l$ in (6). Also when $x = z_1$, $\frac{dy}{dx}$ in (3) must equal $\frac{dy}{dx}$ in (4), and y in (5) must equal y in (6).

If then we make $x = 0$ and $y = 0$ in (5), we find $C_3 = 0$.

If we make $x = l$ and $y = 0$ in (6), we obtain

$$C_4 + C_2l = -\frac{Wz_1l^2}{3}.$$

If we make $x = z_1$ in (3) and (4) and place the two values of $\frac{dy}{dx}$ equal, we have

$$C_1 - C_2 = \frac{Wz_1^2}{2}.$$

If we make $x = z_1$ in (5) and (6) and place the two values of y equal, we have

$$(C_1 - C_2)z_1 - C_4 = \frac{Wz_1^3}{3}.$$

Hence we find, for the constants of integration,

$$C_1 = \frac{Wz_1^2}{2} - \frac{Wz_1 l}{3} - \frac{Wz_1^3}{6l}, \quad C_2 = -\frac{Wz_1 l}{3} - \frac{Wz_1^3}{6l}, \quad C_3 = 0, \quad C_4 = \frac{Wz_1^3}{6}.$$

Substituting these, we obtain

$$\text{for } x < z_1 \qquad EI\frac{dy}{dx} = \frac{W(l - z_1)}{6l}(3x^2 - 2lz_1 + z_1^2); \quad . \quad . \quad . \quad (7)$$

$$\text{for } x > z_1 \qquad EI\frac{dy}{dx} = \frac{Wz_1}{6l}(6lx - 3x^2 - 2l^2 - z_1^2); \quad . \quad . \quad . \quad (8)$$

$$\text{for } x < z_1 \qquad EIy = \frac{W(l - z_1)x}{6l}(x^2 - 2lz_1 + z_1^2); \quad . \quad . \quad . \quad (9)$$

$$\text{for } x > z_1 \qquad EIy = \frac{Wz_1(l - x)}{6l}(x^2 - 2lx + z_1^2). \quad . \quad . \quad . \quad (10)$$

If we make $x = z_1$ in (9) or (10), we have for the deflection Δ_w at the load

$$\Delta_w = -\frac{Wz_1^2 z_2^2}{3EIl},$$

where z_1 and z_2 are the distances of the load from the right and left ends. The deflection at the load is evidently a maximum when $z_1 = z_2 = \frac{l}{2}$, that is, when the load is at the middle of the span. In this case the tangent at the middle is horizontal. When the load is not at the centre of the span, the maximum deflection will evidently be at the same point C in the figure between the load and the *farthest end*.

Let the distance of this point from the left end be m. If then z_1 is less than $\frac{l}{2}$, m is greater than z_1. If z_1 is greater than $\frac{l}{2}$, m is less than z_1. If then we put $\frac{dy}{dx}$ in (8) equal to zero, we have for the distance m from the left end to the point C at which the deflection is a maximum,

$$\text{when } z_1 < \frac{l}{2} \qquad m = l - \sqrt{\frac{1}{3}(2l - z_2)z_2}. \quad . \quad . \quad . \quad . \quad (11)$$

If we put $\frac{dy}{dx}$ in (7) equal to zero, we find for the value of x which makes the deflection a maximum,

$$\text{when } z_1 > \frac{l}{2} \qquad m = \sqrt{\frac{1}{3}(2l - z_1)z_1}. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The distance $l - m$ from the right end in this case is the same as the distance from the left end in the first case, if z_1 in (12) is taken equal to z_2 in (11).

If we substitute the value of m in (12) in the place of x in (9), or the value of m in (11) in the place of x in (10), we have for the maximum deflection,

$$\text{when } z_1 > \frac{l}{2} \qquad \Delta = -\frac{Wz_1 z_2(2l - z_1)}{27EIl}\sqrt{3z_1(2l - z_1)}; \quad . \quad . \quad (13)$$

$$\text{when } z_1 < \frac{l}{2} \qquad \Delta = -\frac{Wz_1 z_2(2l - z_2)}{27EIl}\sqrt{3z_2(2l - z_2)}. \quad . \quad . \quad (14)$$

If the load W is at the middle of the span, $z_1 = z_2 = \frac{l}{2}$, and from (7) and (9) we have for any point between the left end and the centre

$$EI\frac{dy}{dx} = \frac{W}{4}\left(x^2 - \frac{l^2}{4}\right); \quad \ldots \ldots \quad (15)$$

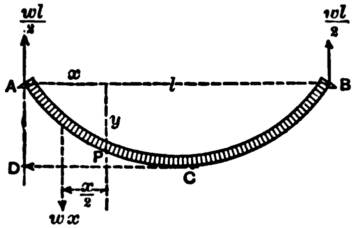
$$EIy = \frac{Wx}{12}\left(x^2 - \frac{3l^2}{4}\right). \quad \ldots \ldots \quad (16)$$

The maximum deflection is at the centre and equal to

$$\Delta = -\frac{Wl^3}{48EI}, \quad \ldots \ldots \quad (17)$$

or only $\frac{1}{16}$ as much as for a beam of the same length fixed at one end and loaded at the other.

Case 4. Horizontal Beam—Uniformly Distributed Load—Constant Cross-section.—Let w be the load per unit of length uniformly distributed. Take the origin at the left end A.



Then the reaction at each end is $\frac{wl}{2}$; and since we can take the load wx as acting at its centre of mass or at a distance of $\frac{x}{2}$ from any point P of the neutral axis, we have for the bending moment at that point

$$M_x = -\frac{wl}{2}x + wx \times \frac{x}{2}.$$

Hence, from (I), page 326,

$$-M_x = EI\frac{d^2y}{dx^2} = \frac{wlx}{2} - \frac{wx^2}{2}. \quad \ldots \ldots \quad (1)$$

If the cross-section is constant, I is constant. For $x = 0$ we must have $y = 0$, and for $x = \frac{l}{2}$ we must have $\frac{dy}{dx} = 0$, since the curve passes through A and B and the tangent is horizontal at the centre C. Determining the constants of integration by these conditions, we have, by integrating (1),

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} - \frac{wl^3}{24}. \quad \ldots \ldots \quad (2)$$

Integrating (2), we have

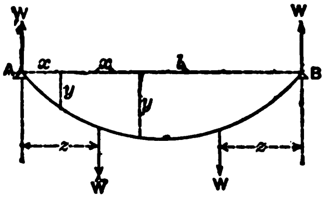
$$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24}. \quad \ldots \ldots \quad (3)$$

The maximum deflection Δ occurs at the centre for $x = \frac{l}{2}$; hence

$$\Delta = -\frac{5wl^4}{384EI},$$

or only $\frac{5}{128}$ of a beam of the same length fixed at one end and uniformly loaded. (For breaking weight see page 305.)

Case 5. Horizontal Beam Supported at Ends—Constant Cross-section—With Two Equal Symmetrically Placed Loads.—Let the beam AB of length l support two loads W, W placed at equal distances z, z from the ends.



The reaction at each support is then W, and the maximum moment is at the centre and equal to Wz.

For the *breaking weight*, then, we have

$$Wz = \frac{S_r I}{v}, \quad \text{or} \quad W = \frac{S_r I}{vz},$$

where S_r is the coefficient of rupture (page 288).

We have from (I), page 326,

$$\text{for } x < z \qquad -M_x = EI\frac{d^2y}{dx^2} = Wx; \quad \ldots \ldots \quad (1)$$

$$\text{for } x > z \qquad -M_x = EI\frac{d^2y}{dx^2} = Wz. \quad \ldots \ldots \quad (2)$$

If the cross-section is constant, I is constant. Since the curve passes through A and B and is horizontal at the centre, we have $y = 0$ for $x = 0$ and $\frac{dy}{dx} = 0$ for $x = \frac{l}{2}$. Hence, integrating (1), we have

$$\text{for } x < z \qquad EI\frac{dy}{dx} = \frac{Wx^2}{2} + C_1. \quad \ldots \ldots \quad (3)$$

Integrating (2), we have

$$\text{for } x > z \qquad EI\frac{dy}{dx} = Wzx - \frac{Wzl}{2}. \quad \ldots \ldots \quad (4)$$

Integrating again, we obtain from (3),

$$\text{for } x < z \qquad EIy = \frac{Wx^3}{6} + C_1 x, \quad \ldots \ldots \quad (5)$$

and from (4),

$$\text{for } x > z \qquad EIy = \frac{Wzx^2}{2} - \frac{Wzlx}{2} + C_2. \quad \ldots \ldots \quad (6)$$

When $x = z$, $\frac{dy}{dx}$ in (3) and (4) must be equal. Hence we have

$$C_1 = \frac{Wz^2}{2} - \frac{Wzl}{2}.$$

Also, when $x = z$, y in (5) and (6) must be equal. Hence we have

$$C_2 = \frac{Wz^3}{6}.$$

Substituting these values of the constants of integration, we have

$$\text{for } x < z \qquad EI\frac{dy}{dx} = \frac{W}{2}(x^2 - lz + z^2); \quad \ldots \ldots \quad (7)$$

$$EIy = \frac{Wx}{6}(x^2 - 3lz + 3z^2); \quad \ldots \ldots \quad (8)$$

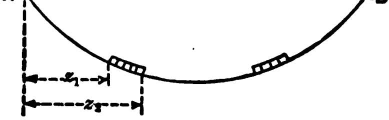
$$\text{for } x > z \qquad EI\frac{dy}{dx} = \frac{Wz}{2}(2x - l), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$EIy = \frac{Wz}{6}(3x^2 - 3lx + z^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The maximum deflection is at the centre and equal to

$$\Delta = -\frac{Wz}{24EI}(3l^2 - 4z^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If the loads are uniformly distributed over the distance $z_2 - z_1$, instead of being concentrated, we can put wdz in place of W. Equation (11) then becomes



$$\Delta = -\int \frac{wzdz}{24EI}(3l^2 - 4z^2).$$

If we integrate this between the limits z_2 and z_1, we have for the deflection at the centre

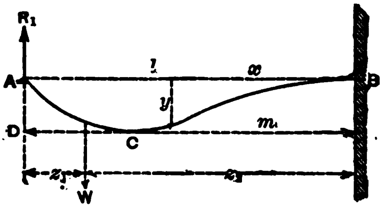
$$\Delta = -\frac{w}{48EI}\left[3l^2(z_2^2 - z_1^2) - 2(z_2^4 - z_1^4)\right]. \quad . \quad . \quad . \quad . \quad (12)$$

If the load covers the whole beam, $z_2 = \frac{l}{2}$, $z_1 = 0$, and we have

$$\Delta = -\frac{5wl^4}{384EI},$$

as already found.

Case 6. Horizontal Beam Fixed at one End and Supported at the Other—Constant Cross-section—Concentrated Load.—Let l be the length of the beam, z_1 the distance of the load W from the supported end, z_2 from the fixed end. Take the origin at the fixed end and let R_1 be the reaction at the supported end A.



Then from (I), page 326, we have

$$\text{for } x > z_2 \qquad -M_x = EI\frac{d^2y}{dx^2} = R_1(l - x); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{for } x < z_2 \qquad -M_x = EI\frac{d^2y}{dx^2} = R_1(l - x) - W(z_2 - x). \quad . \quad . \quad (2)$$

For constant cross-section I is constant. Integrating (1), we have

$$\text{for } x > z_2 \qquad EI\frac{dy}{dx} = R_1lx - \frac{R_1x^2}{2} + C_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Integrating (2), we have

$$\text{for } x < z_2 \quad EI\frac{dy}{dx} = R_1lx - \frac{R_1x^2}{2} - Wz_2x + \frac{Wx^2}{2} + C_2. \quad . \quad . \quad . \quad (4)$$

Integrating again, we obtain from (3),

$$\text{for } x > z_2 \quad EIy = \frac{R_1lx^2}{2} - \frac{R_1x^3}{6} + C_1x + C_3. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and from (4)

for $x < z_2$

$$EIy = \frac{R_1 l x^2}{2} - \frac{R_1 x^3}{6} - \frac{W z_2 x^2}{2} + \frac{W x^3}{6} + C_3 x + C_4. \quad . \quad . \quad . \quad (6)$$

The curve APB must pass through A and B, have a horizontal tangent at B, and each portion from A to W and W to B must have a common tangent and deflection at the load W.

Hence we must have $y = 0$ for $x = 0$ in (6) and $x = l$ in (5). Also $\frac{dy}{dx} = 0$ for $x = 0$ in (4); and when $x = z_2$, $\frac{dy}{dx}$ in (3) must equal $\frac{dy}{dx}$ in (4), and y in (5) must equal y in (6).

If then we make $x = 0$ and $\frac{dy}{dx} = 0$ in (4), we have $C_3 = 0$; and if we make $x = 0$ and $y = 0$ in (6), we have $C_4 = 0$. If we make $x = l$ and $y = 0$ in (5), we have

$$C_2 + C_1 l = -\frac{R_1 l^3}{3}.$$

If we make $x = x_2$ in (3) and (4) and place the two values of $\frac{dy}{dx}$ equal, we have

$$C_1 = -\frac{W z_2^2}{2}.$$

If we make $x = z_2$ in (5) and (6) and place the two values of y equal, we have

$$C_1 z_2 + C_2 = -\frac{W z_2^3}{3}.$$

We have then

$$C_2 = +\frac{W z_2^3}{6} \quad \text{and} \quad R_1 = \frac{W z_2^2}{2l^3}(3l - z_2).$$

Substituting these values, we have

$$\text{for } x > z_2 \qquad EI\frac{dy}{dx} = \frac{W z_2^2}{4l^3}[(2lx - x^2)(3l - z_2) - 2l^3]; \quad . \quad . \quad . \quad (7)$$

$$\text{for } x < z_2 \qquad EI\frac{dy}{dx} = \frac{Wx}{4l^3}[z_2^2(2l - x)(3l - z_2) - 2l^3(2z_2 - x)]; \quad . \quad (8)$$

$$\text{for } x > z_2 \qquad EIy = \frac{W z_2^2}{12l^3}[(3lx^2 - x^3)(3l - z_2) - 2l^3(3x - z_2)]; \quad (9)$$

$$\text{for } x < z_2 \qquad EIy = \frac{W x^2}{12l^3}[z_2^2(3l - x)(3l - z_2) - 2l^3(3z_2 - x)]. \quad (10)$$

If we make $x = z_2$ in (9) or (10), we have for the deflection Δ_w at the load

$$\Delta_w = \frac{W z_2^3}{12EIl^3}[(3l - z_2)^2 z_2 - 4l^3],$$

where z_2 is the distance of the load *from the fixed end.* This deflection at the load is a maximum when $z_2 = l(2 - \sqrt{2})$.

If z_2 is greater than this, the maximum deflection will be at some point C in the figure between the load and the fixed end. If z_2 is less than this,

the point C will be between the load and the supported end. Let the distance of this point from the fixed end be m. If then we put $\frac{dy}{dx}$ in (7) and (8) equal to zero, we have for the distance m *from the fixed end* to the point C at which the deflection is a maximum,

$$\text{when } z_1 < l(2-\sqrt{2}) \qquad m = l - l\sqrt{\frac{l-z_1}{3l-z_1}}; \quad \ldots \ldots \quad (11)$$

$$\text{when } z_1 > l(2-\sqrt{2}) \qquad m = \frac{2lz_1(2l-z_1)}{2l^2 + z_1(2l-z_1)}. \quad \ldots \ldots \quad (12)$$

If we substitute these values of m in the place of x in (9) and (10), we have for the maximum deflection

$$\text{when } z_1 < l(2-\sqrt{2}) \qquad \Delta = -\frac{Wz_1^2}{6EI}(l-z_1)\sqrt{\frac{l-z_1}{3l-z_1}}; \quad \ldots \quad (13)$$

$$\text{when } z_1 > l(2-\sqrt{2}) \qquad \Delta = -\frac{Wz_1^2(l-z_1)(2l-z_1)^2}{3EI[2l^2 + z_1(2l-z_1)]^2}. \quad \ldots \quad (14)$$

These values of Δ are themselves a maximum and equal when

$$z_1 = l(2-\sqrt{2}) = 0.58575l.$$

The greatest possible deflection is then at the load when the load is at a distance of about $0.586l$ from the fixed end.

This greatest possible deflection is

$$\Delta = -\frac{47094\,Wl^3}{4800000EI},$$

or only about $\frac{47}{100}$ as much as for a beam supported at both ends.

If the load is at the middle of the span, we have $R_1 = \frac{5}{16}W$, instead of $\frac{1}{2}W$ as it would be for a beam supported at the ends; and since in this case $z_1 = \frac{1}{2}l < l(2-\sqrt{2})$, we have, from (11) and (13), the maximum deflection at a distance from the fixed end $x = \frac{6}{11}l$, and equal to

$$\Delta = -\frac{Wl^3}{48\sqrt{5}EI},$$

or only $\frac{1}{\sqrt{5}}$ as much as for beam supported at the ends.

There is evidently a point between the load and the fixed end for which the moment is zero.

This is the *point of inflection*. At this point the curve changes from concave to convex. If we put equation (2) equal to zero, and insert the value of R_1, we obtain for the distance of the point of inflection from the fixed end

$$x = \frac{lz_1(2l-z_1)}{2l^2 + 2lz_1 - z_1^2}. \quad \ldots \ldots \ldots \quad (15)$$

If the load is at the centre of the span, this becomes $\frac{8}{11}l$.

Breaking Weight.—Rupture will occur where the moment is greatest, that is, either at the load or at the fixed end.

The moment at the load is, from the figure page 397,

$$-R_1z_1 = R_1z_2 - R_1l.$$

The moment at the fixed end is

$$Wz_2 - R_1l.$$

Now W is always greater than R_1, and hence Wz_2 is greater than R_1z_2. The moment is therefore greatest at the fixed end.

Inserting the value of R_1, we have for the moment at the fixed end

$$Wz_2 = -\frac{Wz_2^3}{2l^3}(3l - z_2) = \frac{S_rI}{v},$$

where S_r is the coefficient of rupture and v the distance of the most remote fibre from the neutral axis. Hence the breaking weight in general is

$$W = \frac{2S_rIl^3}{vz_2(2l - z_2)(l - z_2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The moment at the fixed end is a maximum for

$$z_2 = l\left(1 - \sqrt{\frac{1}{3}}\right) = 0.4226l.$$

This maximum moment at the fixed end is then

$$\frac{Wl}{3\sqrt{3}} = \frac{S_rI}{v},$$

and the least breaking weight is then

$$W = \frac{3\sqrt{3}S_rI}{vl},$$

or $\frac{3\sqrt{3}}{4} = 1.3$ times as great as for beam supported at the ends.

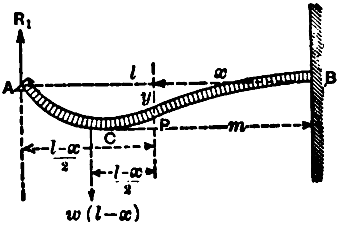
If the load is at the centre of the span, $z_2 = \frac{1}{2}l$ and the breaking weight is

$$W = \frac{16S_rI}{3vl},$$

or $\frac{4}{3}$ as much as for beam supported at the ends.

Case 7. Horizontal Beam—Fixed at One End and Supported at the Other—Constant Cross-section—Load Uniformly Distributed.—Let l be the length of the beam, take the origin at the fixed end, and let R_1 be the reaction at the supported end and w the load per unit of length.

Then from (I), page 326, we have, since we can take the load $w(l - x$ as acting at its centre of mass, or at a distance $\frac{l - x}{2}$ from P,



$$-M_x = EI\frac{d^2y}{dx^2} = R_1(l-x) - \frac{w(l-x)^2}{2}. \quad (1)$$

For constant cross-section I is constant.

Since the curve passes through A and B and the tangent is horizontal at B, we must have $y = 0$ when $x = 0$ and $x = l$, and $\frac{dy}{dx} = 0$ when $x = 0$.

The constants of integration are therefore zero, and we have by integrating (1)

$$EI\frac{dy}{dx} = R_1 lx - \frac{R_1x^2}{2} - \frac{wl^2x}{2} + \frac{wlx^2}{2} - \frac{wx^3}{6} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Integrating (2), we obtain

$$EIy = \frac{R_1lx^2}{2} - \frac{R_1x^3}{6} - \frac{wl^2x^2}{4} + \frac{wlx^3}{6} - \frac{wx^4}{24} \quad . \quad . \quad . \quad . \quad (3)$$

Since for $x = l$, $y = 0$, we have from (3)

$$R_1 = \frac{3}{8}wl,$$

instead of $\frac{1}{2}wl$ as it would be for a beam supported at the ends.

Inserting this value of R_1 in (2) and (3), we have

$$EI\frac{dy}{dx} = -\frac{wx}{48}(6l^2 - 15lx + 8x^2); \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$EIy = -\frac{wx^2}{48}(l - x)(3l - 2x). \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Putting (4) equal to zero, we have for the distance of the point C from the fixed end at which the deflection is a maximum

$$m = \frac{15 - \sqrt{33}}{16}l, \quad \text{or} \quad m = 0.5785l.$$

The maximum deflection itself is then

$$\Delta = -\frac{39 + 55\sqrt{33}}{16^4}\frac{wl^4}{EI}.$$

If we put (1) equal to zero, and insert the value of R_1, we have for the distance of the point of inflection from the fixed end

$$x = \frac{1}{4}l.$$

Breaking Weight.—If we insert the value of R_1 in (1), we have for the moment at any point

$$-M_x = -\frac{wl^2}{8} + \frac{wx}{8}(5l - 4x).$$

This is a maximum when $x = 0$. The maximum moment is then $\frac{wl^2}{8}$ at the fixed end. We have then

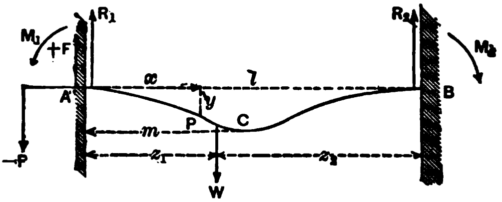
$$\frac{wl^2}{8} = \frac{S_rI}{v},$$

or the breaking weight

$$wl = \frac{8S_rI}{vl},$$

or $\frac{3}{8}$ as great as for the same load in the centre, and just the same as for beam of same length and load supported at the ends.

Case 8. Horizontal Beam Fixed at Both Ends—Constant Cross-section—Concentrated Load.—Let l be the length of beam, z_1 the distance of the load W from the left end, z_2 from the right end. Take the origin at the left end and let R_1 be the reaction at the left end.



The left end must be fixed by a couple $+F$, $-F$ whose moment M_1 is the same at every point of the beam.

Then from (1) page 326, we have

$$\text{for } x < z_1 \qquad -M_x = EI\frac{d^2y}{dx^2} = R_1x - M_1; \quad \dots \quad (1)$$

$$\text{for } x > z_1 \qquad -M_x = EI\frac{d^2y}{dx^2} = R_1x - W(x - z_1) - M_1. \quad \dots \quad (2)$$

For constant cross-section I is constant.

Integrating (1), we have

$$\text{for } x < z_1 \qquad EI\frac{dy}{dx} = \frac{R_1x^2}{2} - M_1x + C_1. \quad \dots \quad (3)$$

Integrating (2), we have

$$\text{for } x > z_1 \qquad EI\frac{dy}{dx} = \frac{R_1x^2}{2} - \frac{Wx^2}{2} + Wz_1x - M_1x + C_2. \quad \dots \quad (4)$$

Integrating again, we obtain from (3)

$$\text{for } x < z_1 \qquad EIy = \frac{R_1x^3}{6} - \frac{M_1x^2}{2} + C_1x + C_3, \quad \dots \quad (5)$$

and from (4)

$$\text{for } x > z_1 \qquad EIy = \frac{R_1x^3}{6} - \frac{Wx^3}{6} + \frac{Wz_1x^2}{2} - \frac{M_1x^2}{2} + C_2x + C_4. \quad (6)$$

The curve APB must pass through A and B, the tangent must be horizontal at A and B, and each portion from A to the load and from B to the load must have a common tangent and deflection at the load. Hence we must have $y = 0$ for $x = 0$ in (5) and $x = l$ in (6). Also we must have $\frac{dy}{dx} = 0$ for $x = 0$ in (3) and $x = l$ in (4); and when $x = z_1$, $\frac{dy}{dx}$ in (3) must equal $\frac{dy}{dx}$ in (4), and y in (5) must equal y in (6).

We have then, making $x = 0$ in (3) and (5), $C_1 = 0$ and $C_3 = 0$. Making $x = z_1$ in (3) and (4) and equating them, we have $C_2 = -\frac{Wz_1^2}{2}$. Making $x = z_1$ in (5) and (6) and equating, we have $C_4 = \frac{Wz_1^3}{6}$. Making $x = l$ and $y = 0$ in (6) and inserting the values of C_2 and C_4, we obtain

$$3M_1l^2 = 3Wl^2z_1 - 3Wlz_1^2 + R_1l^3 - Wl^3 + Wz_1^3.$$

Making $x = l$ and $\frac{dy}{dx} = 0$ in (4) and inserting the value of C_2 we have

$$2M_1 l = 2Wlz_1 - Wz_1^2 + R_1 l^2 - Wl^2.$$

Eliminating M_1 and R_1 from these equations, we obtain

$$R_1 = \frac{Wz_2^2(3z_1 + z_2)}{l^3}, \quad R_2 = \frac{Wz_1^2(3z_2 + z_1)}{l^3};$$

$$M_1 = \frac{Wz_1z_2^2}{l^2}, \quad M_2 = -\frac{Wz_2z_1^2}{l^2}.$$

Substituting these values of R_1 and M_1 and also the values of the constants of integration in equations (3) to (6), we have

$$\text{for } x < z_1 \quad EI\frac{dy}{dx} = \frac{Wz_2^2x}{2l^3}\Big[(3z_1 + z_2)x - 2z_1 l\Big]; \quad \ldots \quad (7)$$

$$\text{for } x > z_1 \quad EI\frac{dy}{dx} = \frac{W}{2l^3}\Big[(3z_1 + z_2)z_2^2x^2 - l^3(x - z_1)^2 - 2z_1z_2^2lx\Big]; \quad (8)$$

$$\text{for } x < z_1 \quad EIy = \frac{Wz_2^2x^2}{6l^3}\Big[(3z_1 + z_2)x - 3z_1 l\Big]; \quad \ldots \quad (9)$$

$$\text{for } x > z_1 \quad EIy = \frac{W}{6l^3}\Big[(3z_1 + z_2)z_2^2x^3 - l^3(x - z_1)^3 - 3z_1z_2^2lx^2\Big]. \quad (10)$$

If we make $x = z_1$ in (9) or (10), we have for the deflection $\varDelta_w$ at the load

$$\varDelta_w = -\frac{Wz_2^3z_1^3}{3l^3EI},$$

where z_1 and z_2 are the distances of the load from the right and left ends.

The deflection at the load is evidently a maximum when $z_1 = z_2 = \frac{l}{2}$, or when the load is at the middle of the span. If the load is not at the centre of the span, the maximum deflection will be at some point C in the figure between the load and the *farthest end*. Let the distance of this point from the left end be m. If then z_1 is greater than $\frac{l}{2}$, m is less than z_1; and if z_1 is less than $\frac{l}{2}$, m is greater than z_1. If then we put $\frac{dy}{dx}$ in (7) and (8) equal to zero, we have for the distance m from the left end to the point C at which the deflection is a maximum,

$$\text{when } z_1 > \frac{l}{2} \quad m = \frac{2z_1 l}{3z_1 + z_2}; \quad \ldots \quad (11)$$

$$\text{when } z_1 < \frac{l}{2} \quad m = \frac{z_2 l}{3z_2 + z_1}. \quad \ldots \quad (12)$$

If we substitute these values of m in the place of x in (9) and (10), we have for the maximum deflection

$$\text{when } z_1 > \frac{l}{2} \quad \varDelta = -\frac{2Wz_1^3z_2^2}{3(3z_1 + z_2)^2EI}; \quad \ldots \quad (13)$$

$$\text{when } z_1 < \frac{l}{2} \quad \varDelta = -\frac{2Wz_2^3z_1^2}{3(3z_2 + z_1)^2EI}. \quad \ldots \quad (14)$$

These values of Δ are themselves a maximum and equal when $z_1 = z_2 = \frac{l}{2}$. The greatest possible deflection is then at the load when the load is in the centre and equal to

$$\Delta = -\frac{Wl^3}{192EI},$$

or only one fourth as much as for a beam supported at both ends.

If we put (1) and (2) equal to zero, we have for the distances of the points of inflection

$$x = \frac{z_1 l}{3z_1 + z_2} \quad \text{and} \quad x = \frac{2z_2 l}{3z_2 + z_1}. \quad . \quad . \quad . \quad . \quad (15)$$

If the load is at the centre, we have

$$x = \frac{1}{4}l \qquad \text{and} \quad x = \frac{3}{4}l.$$

Breaking Weight.—We easily find the greatest moment to be at the end nearest the load and equal to

$$\frac{Wz_1 z_2^2}{l^2} = \frac{S_r I}{v},$$

where z_1 is the distance from the load to the *nearest end.*

Hence the breaking weight in general is

$$W = \frac{S_r I l^2}{v z_1 z_2^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The moment at the nearest end is a maximum for $z_1 = \frac{1}{3}l$, and the least breaking weight is then

$$W = \frac{27 S_r I}{4vl},$$

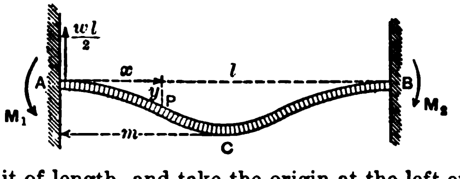
or $\frac{27}{16}$ times as great as for a beam supported at the ends.

If the load is at the centre, we have

$$W = \frac{8 S_r I}{vl},$$

or twice as much as for a beam supported at the end.

Case 9. Horizontal Beam Fixed at Both Ends—Constant Cross-section—Load Uniformly Distributed.—Let l be the length of beam, w



the load per unit of length, and take the origin at the left end. The reaction at each end is evidently $\frac{wl}{2}$. The ends must be fixed by the moments M_1, M_2. We have then from (I), page 326,

$$-M_x = EI\frac{d^2y}{dx^2} = \frac{wl}{2}x - \frac{wx^2}{2} - M_1. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For constant cross-section I is constant. Since for $x = 0$, y and $\frac{dy}{dx}$ are zero, we have, integrating (1),

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} - M_1x; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and integrating (2),

$$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{M_1x^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For $x = l$, $\frac{dy}{dx} = 0$, and we have from (2) and (1)

$$M_1 = +\frac{wl^2}{12}, \qquad M_2 = -\frac{wl^2}{12}.$$

Substituting the value of M_1 in (2) and (3), we have

$$EI\frac{dy}{dx} = -\frac{wx}{12}(l - x)(l - 2x); \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$EIy = -\frac{wx^2}{24}(l - x)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Putting (4) equal to zero, we have for the point C at which the deflection is a maximum, $m = \frac{l}{2}$. The maximum deflection is then

$$\Delta = -\frac{wl^4}{384EI},$$

or only one fifth as much as for the same beam supported at the ends.

If we put (1) equal to zero, we find for the distances of the points of inflection from the origin

$$x = \frac{l}{2} - \frac{l}{2\sqrt{3}}, \qquad x = \frac{l}{2} + \frac{l}{2\sqrt{3}},$$

or $x = 0.2113l$ and $x = 0.7887l$.

Breaking Weight.—The greatest moment is at the fixed ends. Hence

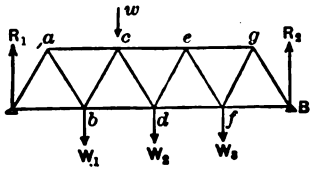
$$\frac{wl^2}{12} = \frac{S_rI}{v},$$

and the breaking weight is

$$wl = \frac{12S_rI}{vl},$$

or $\frac{3}{2}$ as much as for beam supported at the ends.

Deflection of a Framed Structure.—Let a framed structure as shown in the figure be acted upon by the loads W_1, W_2, W_3, applied at the apices b, d, f, and by the reactions R_1 and R_2 at A and B.



Let the deflection Δ at any apex c, loaded or unloaded, be required.

Suppose a load w of any convenient amount placed at that apex. Let the cross-section of any member, as ab, be a, its length l, and its stress due to the total loading, including w, be S. Then its unit stress is $\frac{S}{a}$; and since E is

equal to unit stress divided by unit strain (page 281), its unit strain is $\frac{S}{aE}$ and its entire strain due to the total loading, including w, is $\frac{Sl}{aE}$.

Now let s be the stress in the same member ab, due to w considered as acting alone. Then, since work $= \frac{1}{2}$ stress $\times$ strain, we have for the work on that member due to w alone, $\frac{sSl}{2aE}$.

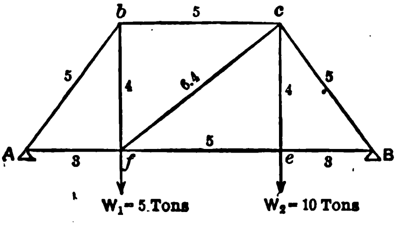
The work on all the members due to w is then $\sum \frac{sSl}{2aE}$. But if Δ is the deflection at c, this work must be equal to $\frac{w\Delta}{2}$. We have then

$$\frac{w\Delta}{2} = \sum \frac{sSl}{2aE}, \quad \text{or} \quad \Delta = \frac{1}{E} \sum \frac{sSl}{waE}.$$

We can thus find the deflection at any apex c, loaded or unloaded. Whatever value we assume for w, the ratio $\frac{s}{w}$ for any member will be the same, since the stress increases with the load. It is therefore convenient to take w unity.

Example.—*Suppose a girder consisting of two inclined rafters Ab and Bc, 5 ft. long, and two vertical ties bf and ce, 4 ft. long; an upper chord bc, 5 ft. long, aud a lower tie consisting of Af, fe and eB, 3 ft., 5 ft. and 3 ft. long respectively. Let there be a diagonal brace fc whose length is 6.4 ft. The loads at f and e are $W_1 = 5$ tons, $W_2 = 10$ tons. Find the deflection at e, taking $E = 12500$ tons per square inch and the area of cross-section of each member as given in the following Table.*

Ans. We easily find (page 106, Example (4)) the stress S in tons in each member due to the total loading, also the stress s in tons in each member due to one ton at e, as given in the Table, (−) signifying compression and (+) tension.



The columns for $\frac{l}{wE}$ and $\frac{sS}{a}$ are then easily filled out. Multiplying these for each member and adding, we find the deflection at e, $\Delta = 0.1627$ inches.

In the same way we could find the deflection at f by supposing $w = 1$ ton at f and placing the corresponding stresses in the fifth column, and the corresponding values of $\frac{Ss}{a}$ in the eighth.

Observe that in such case s for the member cf would be (+) or tension, and $\frac{Ss}{a}$ would be (−), while all other values of $\frac{Ss}{a}$ would be (+). Care should therefore be taken in any case to observe the signs in columns 4, 5 and 8

The stresses S due to total loading are, strictly speaking, slightly changed by the change of shape. This can, however, be disregarded without perceptible error, as the deflection in all practical cases is very small. When it is not, a second approximation can be made by finding s and S for the new shape. The strain due to bending of compressed members is also neglected. The coefficient of elasticity E is assumed constant. All pins, if any, at the apices are presumed to fit tight, and all adjustable members, if any, to be properly adjusted.

A girder after erection may then be tested by calculating the deflection at

the centre for a given loading and comparing with the actual deflection for this loading.

A good agreement is thus a test of the close fit of all pins, of the proper adjustment of all adjustable members, of the agreement of the lengths and the areas of members with those called for by the design, of the constant value

Member.	Length l in inches.	E in tons per square inch.	S in tons.	s in tons.	Area of Cross-section a in sq in.	$\frac{l}{wE}$	$\frac{Ss}{a}$	Δ in inches.
Ab	60	12500	− 7.954	− .341	1.85	$\frac{3}{625}$	+1.46616	0.0371
bc	60	12500	− 4.777	− .2045	1.00	$\frac{3}{625}$	+ .97689	
cB	60	12500	−10.795	− .9091	1.85	$\frac{3}{625}$	+5.30472	
Be	36	12500	+ 6.477	+ .5454	1.5	$\frac{9}{3125}$	+2.355037	0.0979
ef	60	12500	+ 6.481	+ .5454	1.5	$\frac{3}{625}$	+2.356491	
fA	36	12500	+ 4.777	+ .2045	1.5	$\frac{9}{3125}$	+ .651264	
bf	48	12500	+ 6.363	+ .2727	2.0	$\frac{60}{15625}$	+ .867595	0.0277
ce	48	12500	+ 8.636	+1.00	2.0	$\frac{60}{15625}$	+4.318	
cf	76.84	12500	− 2.182	− .4366	0.75	$\frac{76.84}{12500}$	+1.270215	

$\Delta = 0.1627$ inch.

of E and its proper assumption as to magnitude, and finally of the fact that *the limit of elasticity is not exceeded by the loading.*

It is evident that when so many conditions must concur, a discrepancy between the observed and the calculated deflection has little practical significance. The last-mentioned fact, that the limit of elasticity is not exceeded, is the most important, and this is proved, not by close agreement between the actual and the calculated deflections, but by the fact that the deflection is found to remain constant under repeated applications of the loading after the structure has attained its permanent set from the first application. Calculations of deflection are then of little value as a means of testing framed structures.

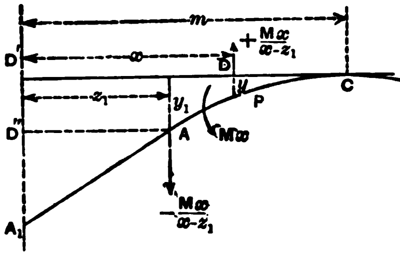
Deflection of Beams found by the Same Principle. — We can make use of the same principle of work in finding the deflection of beams.

Thus let $APCB$ be the curve of the neutral axis of a deflected beam and let the tangent to the curve at the point C be horizontal. Take the origin at any point D' in the horizontal through C. Let z_1, y_1 be the ordinates of the point A *at which curvature begins*, the portion $A'A$, if any, being straight and tangent to the curve ACB at A. Let m be the distance of the point C from the origin, and let x, y be the ordinates of any point P of the curve. Let the moment at P of all the outer forces left or right of P be M_x. We can replace the moment M_x by the couple whose forces $-\frac{M_x}{x - z_1}$ and $+\frac{M_x}{x - z_1}$ act at A and P respectively. The force $+\frac{M_x}{x - z_1}$ at P is the stress which resists deflection at P. Since work is equal to $\frac{1}{2}$ stress × strain (page 281), the work of overcoming this resistance is $\frac{M_x y}{2(x - z_1)}$. Since y is positive above and negative below the horizontal $D'C$ and M_x is positive when counter-clockwise, if we take M_x with a minus sign on the left of P and a plus sign on the right of P we shall always

have $\frac{M_x y}{2(x - z_1)}$ positive. Now the couple whose forces $+\frac{M_x}{x - z_1}$, $-\frac{M_x}{x - z_1}$ act at P and A strains the fibres above and below P in the cross-section at P. We have then

$$\mp \frac{M_x y}{2(x - z_1)} + \left\{ \begin{array}{c} \text{work of straining all} \\ \text{the fibres in the} \\ \text{cross-section at } P \end{array} \right\} = \left\{ \begin{array}{l} \text{total work on all} \\ \text{the fibres between} \\ P \text{ and } C \end{array} \right\}. \quad (A)$$

For any fibre of the cross-section at P at any distance v above or below the neutral axis the unit stress (page 326) is $\frac{M_x v}{I}$. Since E is equal to unit stress divided by unit strain (page 281), the unit strain of the



fibre is $\frac{M_x v}{EI}$. If a is the cross-section of the fibre and dx the distance to the next consecutive cross-section, then $\frac{M_x va}{I}$ is the stress, and $\frac{M_x v dx}{EI}$ is the strain of the fibre between two consecutive cross-sections.

The strain of the fibre limited by the cross-sections at P and C is then

$$\int_x^m \frac{M_x v dx}{EI},$$

and its work is

$$\frac{M_x va}{2I} \int_x^m \frac{M_x v dx}{EI} = \frac{M_x av^2}{2EI^2} \int_x^m M_x dx.$$

Since $\Sigma av^2 = I$, the work of straining all the fibres in the cross-section at P is

$$\frac{M_x}{2EI} \int_x^m M_x dx.$$

Again, the work of straining the fibre between two consecutive cross-sections is

$$\frac{M_x va}{2I} \times \frac{M_x v dx}{EI} = \frac{av^2 M_x^2 dx}{2EI^2}.$$

Since $\Sigma av^2 = I$, the total work on all the fibres between P and C is then

$$\frac{1}{2EI} \int_x^m M_x^2 dx.$$

We have then, from statement (A), for any point P between A and C

$$\mp \frac{M_x y}{2(x - z_1)} = \frac{1}{2EI}\int_x^m M_x^2 dx - \frac{M_x}{2EI}\int_x^m M_x dx.$$

Hence

$$\text{for } x > z_1 \quad EIy = \int_x^m \mp M_x(x - z_1)dx - (x - z_1)\int_x^m \mp M_x dx. \qquad \text{(I)}$$

Differentiating (I), we have

$$EI\frac{dy}{dx} = \mp \int_x^m M_x dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Differentating again,

$$EI\frac{d^2y}{dx^2} = \mp M_x,$$

which is the same as equation (I), page 326.

If in (I) we make $x = z_1$, we have for the deflection at A

$$y_1 = \frac{1}{EI}\int_{z_1}^m \mp M_x(x - z_1)dx,$$

and from (1) for the tangent t_1 of the angle $A'AD''$ which the tangent at A makes with the horizontal

$$t_1 = \mp \int_{z_1}^m M_x dx.$$

We have then for the deflection for any point of the straight portion $A'A$

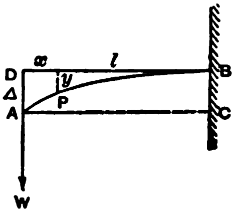
$$y = y_1 + (x - z_1)t_1 = \frac{1}{EI}\int_{z_1}^m \mp M_x(x - z_1)dx - \frac{x - z_1}{EI}\int_{z_1}^m \mp M_x dx,$$

or

$$\text{for } x < z_1 \quad EIy = \int_{z_1}^m \mp M_x(x - z_1)dx - (x - z_1)\int_{z_1}^m \mp M_x dx. \qquad \text{II)}$$

In (I) and (II) M_x is always the moment at any point P of the curve between A, where curvature begins, and C, where the tangent is horizontal. The (—) sign is taken when M_x is taken for all forces on the left, and the (+) sign for all forces on the right.

The application of these equations will give us the same value for the deflection as already obtained.



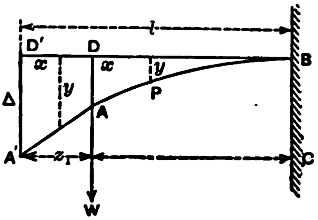
Take the case of the cantilever beam of uniform cross-section fixed horizontally at one end, with load W at the other end. Here $m = l$, $z_1 = 0$, and for W on left of P, $M_x = + Wx$. From (I), then,

$$EIy = \int_x^l - Wx^2dx + x\int_x^l Wxdx.$$

Integrating, we have at once

$$EIy = -\frac{Wl^3}{3} + \frac{Wx^3}{3} + \frac{Wl^2x}{2} - \frac{xW^3}{2} = -\frac{W}{6}(2l^3 - 3l^2x + x^3),$$

which is the same as already found, page 329. If $x = 0$, we obtain for the deflection $\Delta = DA$ at the end



$$EI\Delta = -\frac{Wl^3}{3}.$$

Hence if we take the origin at A, we have

$$EIy = \frac{W}{6}(3l^2x - x^3). \quad . \quad (2)$$

Let the beam project beyond the load W so that the portion $A'A$ is straight, and let the distance of W from A' be z_1. Here $m = l$, and

$$\text{for } W \text{ on left of } P \quad M_x = W(x - z_1).$$

Hence, from (II),

$$\text{for } x < z_1 \left\{ \begin{array}{l} EIy = \int_{z_1}^{l} - W(x - z_1)^2dx + (x - z_1)\int_{z_1}^{l} W(x - z_1)dx; \\ EIy = -\frac{W}{6}\left[2(l - z_1)^3 - 3(l - z_1)^2(x - z_1)\right]. \end{array} \right.$$

If $x = 0$, we obtain for the deflection $\Delta = D'A'$ at the end

$$EI\Delta = -\frac{W}{6}\left[2(l - z_1)^3 + 3(l - z_1)^2 z_1\right].$$

Hence if we take the origin at A', we have

$$\text{for } x < z_1 \; EIy = \frac{W(l - z_1)^2 x}{2}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (I) we have

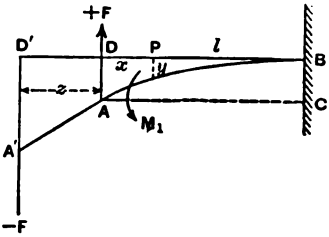
$$\text{for } x > z_1 \left\{ \begin{array}{l} EIy = \int_{x}^{l} - W(x - z_1)^2dx + (x - z_1)\int_{x}^{l} W(x - z_1)dx; \\ EIy = -\frac{W}{6}\left[2(l - z_1)^3 - 3(l - z_1)^2(x - z_1) + (x - z_1)^3\right]. \end{array} \right.$$

Hence if we take the origin at A', we have

$$\text{for } x > z_1 \quad EIy = \frac{W}{6}\left[3(l - z_1)^2x - (x - z_1)^3\right]. \quad . \quad . \quad (4)$$

Let the beam be acted upon by a couple whose forces $+ F$, $- F$ act at A' and A respectively. Take the origin at D.

The moment of a couple is the same at every point in its plane, and equal to $Fz = M_1$. We have then in this case for any point P on the right of D, $M_x = M_1$, and from (I), making $z_1 = 0$, $m = l$,



$$EIy = \int_{x}^{l} - M_1 x dx + x\int_{x}^{l} M_1 dx.$$

$$EIy = -\frac{M_1 l^2}{2} + M_1 lx - \frac{M_1 x^2}{2}.$$

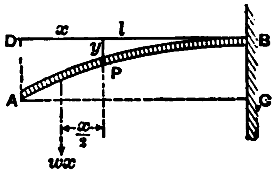
If $x = 0$, we obtain for the deflection $\Delta = DA$

$$EI\Delta = -\frac{M_1 l^2}{2}.$$

Hence if we take the origin at A, we have

$$EIy = M_1lx - \frac{M_1x^2}{2} - \frac{M_1x}{2}(2l - x). \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Let the beam be uniformly loaded with w per unit of length, and take the origin at D. In this case we have for any point P on the right of A, $M = \frac{wx^2}{2}$.



Hence from (I) taking $m = l$ and $z_1 = 0$,

$$EIy = \int_x^l - \frac{wx^3dx}{2} + x\int_x^l \frac{wx^2dx}{2}.$$

$$EIy = -\frac{w}{24}(3l^4 - 4l^3x + x^4),$$

which is the same as already found, page 332.

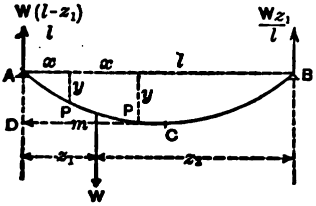
If $x = 0$, we obtain for the deflection $\Delta = DA$,

$$EI\Delta = -\frac{wl^4}{8}.$$

Hence if we take the origin at A, we have

$$EIy = \frac{w}{29}(4l^3x - x^4). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

By using these equations we can find the deflection for all other cases. Thus let a horizontal beam of uniform cross-section have the load W between the supports and take the origin at A. The reaction at A is $\frac{W(l - z_1)}{l}$.



The deflection due to this reaction at any point between A and the point C at which the tangent is horizontal we find from (2), by making $l = m$ and $W = -\frac{W(l - z_1)}{l}$,

$$-\frac{W(l - z_1)}{6EIl}(3m^2x - x^3).$$

The deflection due to W at any point between A and W when $z_1 < m$ we find from (3) by putting $l = m$:

$$\frac{W(m - z_1^2)x}{2EI}.$$

The deflection due to W at any point between W and C when $z_1 < m$ we find from (4) by putting $l = m$:

$$\frac{W}{6EI}\Big[3(m - z_1)^2x - (x - z_1)^3\Big].$$

The deflection due to the reaction $\frac{Wz_1}{l}$ at B at any point between C

and the right end when $z_1 < m$ we find from (2) by putting $l = l - m$, $x = l - x$, and $W = -\frac{Wz_1}{l}$:

$$-\frac{Wz_1}{6EIl}\left[3(l-m)^2(l-x) - (l-x)^3\right].$$

We have then, when $z_1 < m$,

for $x < z_1$

$$EIy = -\frac{W(l-z_1)}{6l}(3m^2x - x^3) + \frac{W(m-z_1)^2x}{2}; \quad \ldots \ldots \quad (7)$$

for $x > z_1$ and $< m$

$$EIy = -\frac{W(l-z_1)}{6l}(3m^2x - x^3) + \frac{W(m-z_1)^2x}{2} - \frac{W}{6}(x-z_1)^3; \quad . \quad (8)$$

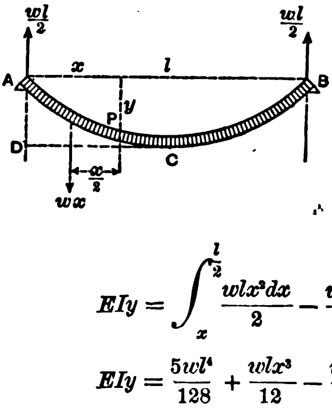
for $x > m$

$$EIy = -\frac{Wz_1}{6l}\left[3(l-m)^2(l-x) - (l-x)^3\right]. \quad \ldots \ldots \quad (9)$$

If we make $x = m$ in (8) and (9) and equate, we obtain

when $z_1 < m$ $\quad m = l - \sqrt{\frac{1}{3}(l^2 - z_1^2)} = l - \sqrt{\frac{1}{3}(2l - z_1)z_1},$

which is the same as already found, page 334.



If we substitute this value of m in (7) and (8), we obtain equations (9) and (10), page 334.

Let the beam sustain a uniformly-distributed load of w per unit of length.

In this case $M_x = -\frac{wl}{2}x + \frac{wx^2}{2}$.

From (I), if we make $z_1 = 0$, $m = \frac{l}{2}$, we have

$$EIy = \int_x^{\frac{l}{2}} \frac{wlx^2dx}{2} - \frac{wx^3dx}{2} - x\int_x^{\frac{l}{2}} \frac{wlxdx}{2} - \frac{wx^2dx}{2};$$

$$EIy = \frac{5wl^4}{128} + \frac{wlx^3}{12} - \frac{wl^3x}{24} - \frac{wx^4}{24}.$$

If $x = 0$, we obtain for the deflection $\Delta = DA$

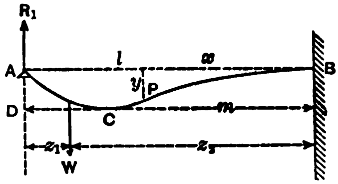
$$EI\Delta = \frac{5wl^4}{28}.$$

Hence, if we take the origin at A, we have

$$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24},$$

which is the same as already found, page 335.

Let the beam be fixed horizontally at one end and supported at the other and sustain the load W at the distance z_1 from the supported end A.



Let R_1 be the reaction at A, and take the origin at the fixed end B.

The deflection due to R_1 at any point between A and B we find from (2), by making $x = l - x$ and $W = -R_1$,

$$-\frac{R_1}{6EI}\left[3l^2(l-x)-(l-x)^3\right].$$

The deflection due to W at any point between A and W we find from (3) by putting $x = l - x$:

$$\frac{W(l-z_1)^2(l-x)}{2EI}.$$

The deflection due to W at any point between W and B we find from (4) by putting $x = l - x$:

$$\frac{W}{6EI}\left[3(l-z_1)^2(l-x)-(l-x-z_1)^3\right].$$

We have then

for $x > z_2$

$$EIy = -\frac{R_1}{6}\left[3l^2(l-x)-(l-x)^3\right]+\frac{W(l-z_1)^2(l-x)}{2}; \quad . \quad . \quad . \quad . \quad . \quad (10)$$

for $x < z_2$

$$EIy = -\frac{R_1}{6}\left[3l^2(l-x)-(l-x^3)\right]+\frac{W(l-z_1)^2(l-x)}{2}-\frac{W}{6}(l-x-z_1)^3. \quad (11)$$

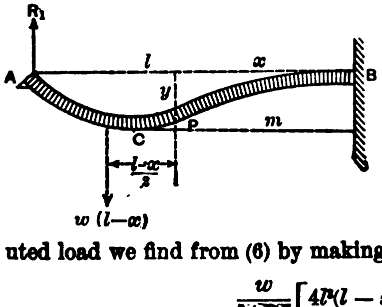
If we make $x = 0$ in (11), $y = 0$, and we obtain

$$R_1 = \frac{Wz_2^2}{2l^3}(3l-z_2),$$

which is the same as already found, page 338.

If we substitute this value of R_1 in (10) and (11), we obtain equations (9) and (10), page 338.

Let the beam be fixed horizontally at one end and supported at the other and uniformly loaded with the load w per unit of length. Take the origin at the fixed end B.



The deflection due to R_1 at any point P we find from (2) by making $x = l - x$ and $W = -R_1$:

$$-\frac{R_1}{6EI}\left[3l^2(l-x)-(l-x)^3\right].$$

The deflection due to the distributed load we find from (6) by making $x = l - x$:

$$\frac{w}{24EI}\left[4l^3(l-x)-(l-x)^4\right].$$

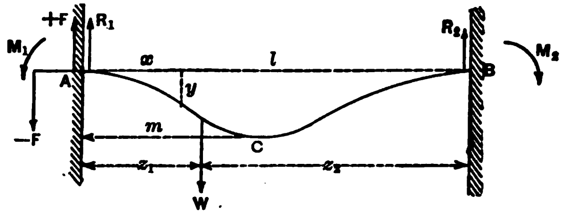
Hence

$$EIy = -\frac{R_1}{6}\left[3l^2(l-x)-(l-x)^3\right]+\frac{w}{24}\left[4l^3(l-x)-(l-x)^4\right]. \quad . \quad (12)$$

For $x = 0$ in (12), $y = 0$, and we find $R_1 = \frac{3}{8}wl$.

Substitute this value of R_1 in (12) and we obtain equation (5), page 341.

Let the beam be fixed horizontally at both ends and have the load W at the distance z_1 from the left end A. Then we have at A the reaction R_1 and the moment M_1.



Take the origin at A.

Then we have for the deflection due to M_1, from (5)

$$\frac{M_1x}{2EI}(2l - x).$$

For the deflection due to R_1 we find from (2) by putting $W = -R_1$:

$$-\frac{R_1}{6EI}[3l^2x - x^3].$$

For the deflection due to W at any point between A and W we find from (3)

$$\frac{W(l - z_1)^2x}{2EI}.$$

For the deflection due to W at any point between W and B we find from (4)

$$\frac{W}{6EI}\left[3(l - z_1)^2x - (x - z_1)^3\right].$$

Hence, for $x < z_1$

$$EIy = -\frac{R_1}{6}\left(3l^2x - x^3\right) + \frac{M_1x}{2}\left(2l - x\right) + \frac{W(l - z_1)^2x}{2}; \quad . \quad . \quad . \quad . \quad (13)$$

for $x > z_1$

$$EIy = -\frac{R_1}{6}\left(3l^2x - x^3\right) + \frac{M_1x}{2}\left(2l - x\right) + \frac{W}{6}\left[3\left(l - z_1\right)^2x - \left(x - z_1\right)^3\right]. \quad (14)$$

Differentiating (13), we have

for $x < z_1$

$$EI\frac{dy}{dx} = -\frac{R_1}{6}\left(3l^2 - 3x^2\right) + \frac{M_1}{2}\left(2l - 2x\right) + \frac{W(l - z_1)^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

For $x = l$, $y = 0$ in (14) and we obtain

$$-\frac{R_1l^3}{3} + \frac{M_1l^2}{2} + \frac{W}{6}(l - z_1)^2(2l + z_1) = 0.$$

For $x = 0$, $\frac{dy}{dx} = 0$ in (15) and we obtain

$$-\frac{R_1l^2}{2} + M_1l + \frac{W(l - z_1)^2}{2} = 0.$$

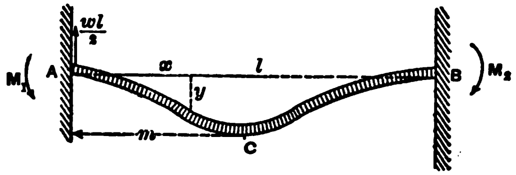
From these two equations we find

$$M_1 = \frac{Wz_1z_2^2}{l^2}, \quad R_1 = \frac{Wz_2^2(3z_1 + z_2)}{l^3},$$

which are the same as already found, page 343.

If we substitute these values of M_1 and R_1 in (13) and (14), we obtain equations (9) and (10), page 343.

Let the beam be fixed horizontally at both ends and be loaded uniformly



with the load w per unit of length. Take the origin at A. Then for the deflection due to M_1 we have from (5)

$$\frac{M_1x}{2EI}(2l - x).$$

For the deflection due to the reaction $\frac{wl}{2}$ at A we have from (2), putting $W = -\frac{wl}{2}$,

$$-\frac{wl}{12EI}[3l^2x - x^3].$$

For the deflection due to the distributed load from (6),

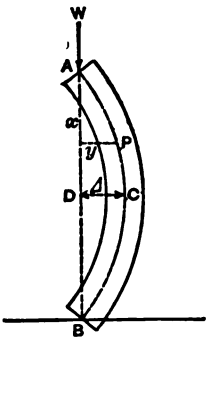
$$\frac{w}{24EI}(4l^3x - x^4).$$

Hence

$$EIy = -\frac{wl}{12}(3l^2x - x^3) + \frac{M_1x}{2}(2l - x) + \frac{w}{24}(4l^3x - x^4),$$

which is the same as equation (3), page 345.

Formulas for Long Struts.—Let a long strut or vertical column of constant cross-section A sustain the load W, and let the deflected column be free to turn at both ends, as in the figure. Take the origin at the upper end A, and let x be the vertical and y the horizontal co-ordinate of any point P of the elastic curve.



Equation (I), page 326, holds for flexure, provided (page 326) that the deflection is small, that a plane section before flexure remains plane after, that the elastic limit is not exceeded and that the coefficient of elasticity E is constant.

The bending moment at the point P is $M_x = Wy$. Hence from equation (I), page 326,

$$EI\frac{d^2y}{dx^2} = -Wy.$$

Multiply both sides of this equation by $2dy$ and we have

$$EI\frac{2dyd^2y}{dx^2} = -2Wydy.$$

Integrating, we have

$$EI\frac{dy^2}{dx^2} = -Wy^2 + C_1.$$

Let $DC = \Delta$ be the maximum deflection. Then when $y = \Delta$, $\frac{dy}{dx}$ is zero, and $C_1 = W\Delta^2$. Hence, substituting this value of C_1, we have by inversion

$$dx = \sqrt{\frac{EI}{W}} \cdot \frac{dy}{\sqrt{\Delta^2 - y^2}}.$$

Integrating again, we have

$$x = \sqrt{\frac{EI}{W}} \text{ arc sin } \frac{y}{\Delta} + C_2.$$

When $y = 0$, x is zero and therefore C_2 is zero. We have then for the equation of the elastic curve

$$y = \Delta \sin x \sqrt{\frac{W}{EI}},$$

which is the equation of a sinuroid. If the length AB of the column is l, then when $x = l$, y is zero. Hence if n is 1, 2, 3, etc., we have

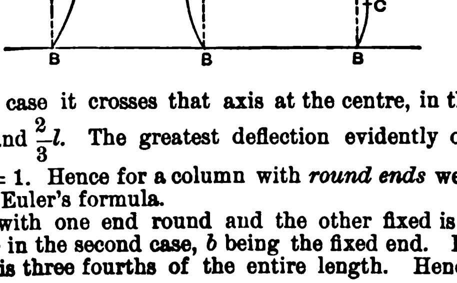
$$l\sqrt{\frac{W}{EI}} = n\pi, \quad \text{or} \quad W = EI\frac{n^2\pi^2}{l^2}.$$

Since $I = A\kappa^2$, where A is the area and κ the radius of gyration of the cross-section for the axis through its centre of mass at right angles to the plane of bending of the axis, we have

$$\frac{W}{A} = \frac{n^2\pi^2 E\kappa^2}{l^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (E)$$

This equation (E) is known as "*Euler's formula*" for long struts.

For $n = 1$, $n = 2$, $n = 3$, we have the curves shown in the following figure. In the first case the curve is entirely on one side of the axis of x,



in the second case it crosses that axis at the centre, in the third case it crosses at $\frac{1}{3}l$ and $\frac{2}{3}l$. The greatest deflection evidently occurs for the case where $n = 1$. Hence for a column with *round ends* we have theoretically $n = 1$ in Euler's formula.

A column with one end round and the other fixed is represented by the portion Ab in the second case, b being the fixed end. Here $n = 2$ and the length Ab is three fourths of the entire length. Hence for a column

with one end fixed and the other round we have theoretically $n = \frac{3}{2}$ in Euler's formula and

$$\frac{W}{A} = \frac{9\pi^2 E\kappa^2}{4l^2}.$$

A column with fixed ends is represented by the portion cc in the third case. Here $n = 3$ and the length cc is three fourths of the entire length. Hence for a column with fixed ends we have theoretically $n = \frac{4}{2} = 2$ in Euler's formula and

$$\frac{W}{A} = \frac{4\pi^2 E\kappa^2}{l^2}.$$

These ideal end conditions do not, however, exist in practice. The nearest approach to round ends is for pins at each end. In such case there is always friction. The nearest approach to a fixed end is a square end abutting upon a rigid base. But since the fibres on the convex side are in tension, the end in this case is only imperfectly fixed.

Practical Values for n. — Brittle materials, such as stone, brick, cement, or hard cast steel, when they fail by crushing, crack and separate into pieces. Tough materials, such as wrought iron, rolled steel, timber, etc., when compressed fail by slow flowing of the material. The crushing load, then, for such materials is the load which produces permanent set. We therefore consider the elastic limit S_e as the "ultimate strength" in such cases. From many experiments carried to the point of failure n in Euler's formula has been found to have the following values:

	Two Pin Ends.	One Pin, One Flat End.	Two Flat Ends.
n	$\sqrt{\frac{5}{3}}$	$\frac{5}{2\sqrt{3}}$	$\sqrt{\frac{5}{2}}$

If then we use these values of n in Euler's formula (E), we obtain for any value of l and κ the so-called "crippling unit load," that is, the unit load $\frac{W}{A}$ which makes the unit stress in the outer fibre of greatest stress equal to the elastic limit S_e when failure occurs.

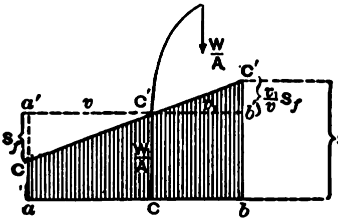
Limiting Length for Euler's Formula.—Let ab represent the cross-section of area A at the centre of the column where the deflection is greatest, C the centre of mass of this cross-section. The plane of bending will always be parallel to the least radius of gyration of the cross-section. Let v and v_1 be the distances parallel to the plane of bending of the axis, of the most remote fibres aa', bb' from the centre C on the convex and concave sides respectively. For symmetrical cross-sections $v = v_1$.

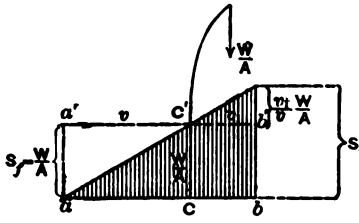
Let S_e be the elastic limit and S_f the unit stress due to bending in the most remote fibre aa' on the convex side. We also have a uniform unit stress of direct compression $\frac{W}{A}$ over the entire cross-section due to the load W. On the convex side this unit stress for the most remote fibre aa' is diminished by the unit stress S_f due to bending. On the concave side

this unit stress for the most remote fibre bb' is increased by the unit stress $\frac{v_1}{v}S_f$ due to bending.

As long as the length l of the column is less than a certain length L, we see from the first figure that when $\frac{W}{A} + \frac{v_1}{v}S_f$ on the concave side equals S_e, the elastic limit, S_f on the convex side will be less than $\frac{W}{A}$ and we shall have compression at every point of the cross-section ab. So long as this is the case Euler's formula (E) does not apply.

But now as the length l increases, we can evidently have a certain





length L for which, when the unit stress on bb' equals the elastic limit S_e, S_f, as shown in the second figure, shall be just equal to $\frac{W}{A}$. When this is the case there is no compression at a. For any length greater than L, then, we shall have tension at a when the unit stress at b is equal to S_e.

At or above the length L, then, Euler's formula applies.

We have for this length the condition

$$\frac{W}{A} + \frac{v_1 W}{vA} = S_e, \quad \text{or} \quad \frac{W}{A} = \frac{S_e}{1 + \frac{v_1}{v}}.$$

But since Euler's formula applies, we have also

$$\frac{W}{A} = \frac{n^2\pi^2 E\kappa^2}{L^2}.$$

Equating these two values of $\frac{W}{A}$, we have for the length L

$$L = \frac{n\kappa\pi\sqrt{\left(1 + \frac{v_1}{v}\right)E}}{\sqrt{S_e}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (L)$$

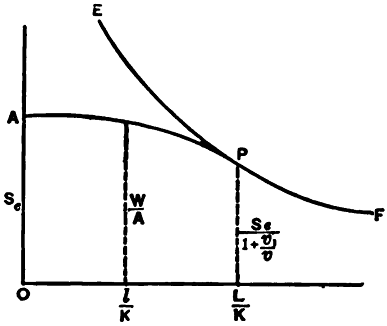
Equation (L) gives then the limiting length above which we can use Euler's formula (E). If the length l is less than L, we cannot use Euler's formula, but must deduce some other formula for the "crippling unit load." The value of κ is always the least radius of gyration of the cross-section.

The Straight-line and Parabola Formulas. — We have seen that for values of $l > L$ we can find the crippling unit load $\frac{W}{A}$ from Euler's formula (E) if we use the values of n given on page 357.

Let us take any origin O and take $x = \frac{l}{\kappa}$ as abscissa and $y = \frac{W}{A}$ as

ordinate. Then Euler's formula is represented by the curve EPF whose equation is

$$y = \frac{n^2\pi^2 E}{x^2}. \quad . \quad . \quad (1)$$



Only the portion PF of this curve can be used, the point P being given by $x = \frac{L}{\kappa}$ and $y = \dfrac{S_e}{1 + \frac{v_1}{v}}$.

For $l < L$ let the curve for the ideal column be AP. The ideal column is perfectly straight, perfectly homogeneous in all its parts, the load W accurately at the centre of cross-section, etc. No column is thus ideally perfect, and hence the actual values of $\frac{W}{A}$ as given by experiment are found distributed above and below AP over a considerable range. Evidently, then, a strictly rational formula for AP would have no advantage over any convenient curve which passes through A and P so that $OA = S_e$ for $l = 0$, and $y = \dfrac{S_e}{1 + \frac{v_1}{v}} = \frac{W}{A}$ for $l = L$, and has at P a common tangent with Euler's curve PE.

Let us assume, then, for the curve AP

$$y = S_e + bx + cx^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This curve passes through A so that $OA = S_e$ for $l = 0$. It remains to determine b and c, so that the curve shall pass through P and have a common tangent at P with Euler's curve.

If we make $x = \frac{L}{\kappa}$ in (1) and (2) and equate, we have for the condition that the curve passes through P

$$S_e + \frac{bL}{\kappa} + \frac{cL^2}{\kappa^2} = \frac{n^2\pi^2 E\kappa^2}{L^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we differentiate (1) and (2) and equate $\frac{dy}{dx}$ in both cases for $x = \frac{L}{\kappa}$, we have for the condition of a common tangent at P

$$b + 2c\frac{L}{\kappa} = -\frac{2n^2\pi^2 E\kappa^3}{L^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From (3) and (4) we obtain

$$b = -\frac{2S_e\kappa}{L} + \frac{4n^2\pi^2 E\kappa^3}{L^3};$$

$$c = \frac{S_e\kappa^2}{L^2} - \frac{3n^2\pi^2 E\kappa^4}{L^4}.$$

Substituting these values of b and c in (2), and putting $x = \frac{l}{\kappa}$ and

$y = \frac{W}{A}$, we obtain for the crippling unit load

$$\text{for } l < L \qquad \frac{W}{A} = S_e\left[1 + \frac{2(v - v_1)l}{(v + v_1)L} - \frac{(2v - v_1)l^2}{(v + v_1)L^2}\right]. \quad . \quad . \quad . \quad \text{(SP)}$$

We call equation (SP) the "straight-line parabola" formula for long struts, because if $v_1 = 2v$, the third term in the parenthesis disappears and the curve AP becomes a straight line, while if $v = v_1$, as is the case for symmetrical cross-sections, the second term disappears and the curve AP becomes a parabola.

We have thus for $v_1 = 2v$ the straight-line formula for crippling load,

$$\text{for } l < L \text{ and } v_1 = 2v \qquad \frac{W}{A} = S_e\left(1 - \frac{2}{3}\frac{l}{L}\right), \quad . \quad . \quad . \quad . \quad . \quad \text{(S)}$$

$$\text{where } L = \frac{n\kappa\pi\sqrt{3E}}{\sqrt{S_e}}.$$

For $v = v_1$ or for symmetrical cross-sections we have the parabola formula for crippling load,

$$\text{for } l < L \text{ and } v = v_1 \qquad \frac{W}{A} = S_e\left(1 - \frac{1}{2}\frac{l^2}{L^2}\right), \quad . \quad . \quad . \quad . \quad . \quad \text{(P)}$$

$$\text{where } L = \frac{n\kappa\pi\sqrt{2E}}{\sqrt{S_e}}.$$

The value of κ is always the least radius of gyration of the cross-section.

Both equations (S) and (P) are well known, and (S) especially has come into very general use. We see that both are special cases of the general formula (SP) here given for the first time.

Rankine-Gordon Formula.—From the figure page 358 we see that when $l < L$ we have

$$\frac{v_1}{v}S_f + \frac{W}{A} = S_e.$$

If we assume that for lengths less than L, S_f increases approximately as the square of the length, we have

$$S_f : \frac{W}{A} :: l^2 : L^2, \quad \text{or} \quad S_f = \frac{Wl^2}{AL^2}.$$

Inserting this value of S_f in the preceding equation, we obtain for the crippling unit load

$$\text{for } l < L \qquad \frac{W}{A} = \frac{S_e}{1 + \frac{v_1 l^2}{vL^2}} = \frac{S_e}{1 + \frac{v_1 S_e l^2}{n^2\pi^2(v + v_1)E\kappa^2}}. \quad . \quad . \quad . \quad \text{(H)}$$

We call equation (H) the "hyperbola formula," because it is the equation of an hyperbola.

Equation (H) is usually given in the form

$$\frac{W}{A} = \frac{S_e}{1 + a\frac{l^2}{\kappa^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(RG)}$$

where a is an experimental constant, and *in this form* it is known as the "Rankine-Gordon formula for long struts." We see that the experimental

constant a really depends upon the end conditions as given by n, upon the values of v and v_1, and upon the ratio of the elastic limit S_e to the coefficient of elasticity E. We see also that (H) must not be used for $l > L$. The curve of (RG) or (H) passes through A (figure page 359), so that $OA = S_e$ for $l = 0$, and also passes through P for $l = L$, but it has not a common tangent at P. Still it gives good results, and in the form (RG) is widely used. Equation (H) is a more general form of the Rankine-Gordon formula here given for the first time.

The value of κ is always the least radius of gyration.

Recapitulation of Formulas for Long Struts.—The straight-line formula (S) and the parabola formula (P) are well known and widely used. As we have seen, they are special cases of the general (SP) formula here given for the first time. The Rankine-Gordon formula (RG) is also a special experimental form of the more general and rational hyperbola formula (H) here given for the first time.

We recapitulate here for convenience of reference all these formulas for long struts.

Let A be the constant area of cross-section, W the crippling load and therefore $\frac{W}{A}$ the crippling unit load which makes the unit stress in the most compressed fibre just equal to the elastic limit S_e.

Let κ be the least radius of gyration of the cross-section for the axis through its centre of mass of right angles to the plane of bending of the axis.

Let v and v_1 be the distances parallel to the plane of bending of the most remote fibres, on the convex and concave sides respectively, from the centre of the cross-section. For symmetrical cross-sections $v = v_1$.

Let n be a number depending on the end conditions, as follows:

	Two Pin Ends.	One Pin, One Flat End.	Two Flat Ends.
n	$\sqrt{\frac{5}{3}}$	$\frac{5}{2\sqrt{3}}$	$\sqrt{\frac{5}{2}}$
$n\pi$	4	4.5	5
$n^2\pi^2$	16	20	25

Then we have for the limiting length L above which Euler's formula holds

$$L = \frac{n\kappa\pi\sqrt{\left(1 + \frac{v_1}{v}\right)E}}{\sqrt{S_e}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (L)$$

Let l be the length of strut. Then we have for the crippling unit load $\frac{W}{A}$ Euler's formula,

$$\text{when } l > L \qquad \frac{W}{A} = \frac{n^2\pi^2 E\kappa^2}{l^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (E)$$

If $l < L$, we may use either the generalized Rankine-Gordon formula,

$$\text{when } l < L \qquad \frac{W}{A} = \frac{S_e}{1 + \frac{v_1 l^2}{vL^2}} = \frac{S_e}{1 + \frac{v_1 S_e l^2}{n^2\pi^2(v + v_1)E\kappa^2}}, \quad . \quad . \quad (H)$$

or the formula (SP),

when $l < L$
$$\frac{W}{A} = S_e\left[1 + \frac{2(v - v_1)l}{(v + v_1)L} - \frac{(2v - v_1)l^2}{(v + v_1)L^2}\right]. \quad . \quad (SP)$$

For $v_1 = 2v$ formula (SP) becomes the "straight-line" formula,

when $l < L$
$$\frac{W}{A} = S_e\left[1 - \frac{2}{3}\frac{l}{L}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (S)$$

$$\text{where } L = \frac{n\kappa\pi\sqrt{3E}}{\sqrt{S_e}}.$$

For $v = v_1$ or for symmetrical cross-sections formula (SP) becomes the "parabola" formula,

when $l < L$
$$\frac{W}{A} = S_e\left[1 - \frac{1}{2}\frac{l^2}{L^2}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (P)$$

$$\text{where } L = \frac{n\kappa\pi\sqrt{2E}}{\sqrt{S_e}}.$$

In all cases we must divide the crippling load by the factor of safety assumed (page 291), in order to obtain the safe load; or we can replace S_e in formulas (P), (S) and (SP), and *in the numerator* of G, by the value of S_w as determined page 292.

For the average values of S_e and E given in our Table page 823, we obtain from (L) the following values of $\frac{L}{\kappa}$.

	S_e Lbs. per square in.	E Lbs. per square in.	$\frac{E}{S_e}$	Value of $\frac{L}{\kappa}$ when $v = v_1$. Two Pin Ends.	One Pin, One Flat End.	Two Flat Ends.
Wrought iron...	25000	25000000	1000	180	200	220
Steel...........	40000	30000000	750	150	170	190
Cast iron.......	60000	15000000	250	90	100	110
Timber.........	3000	15000000	500			160

	Value of $\frac{L}{\kappa}$ in general. Two Pin Ends.	One Pin, One Flat End.	Two Flat Ends.
Wrought iron.....	$120\sqrt{1+\frac{v_1}{v}}$	$141\sqrt{1+\frac{v_1}{v}}$	$158\sqrt{1+\frac{v_1}{v}}$
Steel............	$109\sqrt{1+\frac{v_1}{v}}$	$122\sqrt{1+\frac{v_1}{v}}$	$136\sqrt{1+\frac{v_1}{v}}$
Cast iron.........	$68\sqrt{1+\frac{v_1}{v}}$	$71\sqrt{1+\frac{v_1}{v}}$	$79\sqrt{1+\frac{v_1}{v}}$
Timber..........			$112\sqrt{1+\frac{v_1}{v}}$

In practice $\frac{l}{\kappa}$ is usually less than 100, so that formula (H) or (SP) covers the range of ordinary practice, and we seldom have to use formula (E).

EXAMPLES.

(1) *A cylindrical beam 2 inches in diameter, 60 inches long and weighing $\frac{1}{4}$ lb. per cubic inch deflects $\frac{3}{8}$ inch under a weight of 3000 lbs. at the centre. Find E.*

Ans. $E = 28929144$ lbs. per square inch.

(2) *A rectangular beam 5 ft. long, 3 inches wide and 3 inches deep is deflected $\frac{1}{16}$ inch by a weight of 3000 lbs. applied at the centre. Find E.*

Ans. $E = 20000000$ lbs. per square inch.

(3) *A beam whose length is 16 ft., width 2 inches, depth 12 inches, and coefficient of elasticity 16000000 lbs. is deflected half an inch by a weight at the centre. Find the weight, neglecting the weight of the beam.*

Ans. Weight = 1562 lbs.

(4) *An iron rectangular beam whose length is 12 ft., breadth $1\frac{1}{2}$ in., coefficient of elasticity 24000000 lbs. has a weight of 10000 lbs. suspended at the middle. Find the depth in order that the deflection may be $\frac{1}{480}$ of the length.*

Ans. Depth = 8.8 in.

(5) *A rectangular wooden beam 6 in. wide and 30 ft. long is supported at the ends. The coefficient of elasticity is $E = 1800000$ lbs. per square inch. The weight of a cubic foot of the beam is 50 lbs. Find the depth that it may deflect one inch from its own weight. How deep must it be to deflect $\frac{1}{480}$ of its length?*

Ans. Depth = 6.5 inches; depth = 6.8 inches.

(6) *Required the depth of a rectangular beam which is supported at the ends and so loaded at the middle that the elongation of the lowest fibre shall equal $\frac{1}{1400}$ of its original length.*

Ans. Depth $= \sqrt{\dfrac{2100\,Wl}{Eb}}$.

(7) *Required the radius of curvature at the middle point of a wooden beam when the load is 3000 lbs., the length 10 ft., breadth 4 inches, depth 8 inches and $E = 1000000$ lbs.*

Ans. Radius = 1896 inches.

(8) *Let the beam be of iron supported at the ends. Let the breadth be 1 in., depth 2 in., length 8 ft. and $E = 25000000$ lbs. Required the radius of curvature at the middle when the deflection is $\frac{1}{2}$ inch.*

Ans. Radius = 3840 inches.

(9) *If a beam 6 ft. long, $1\frac{1}{2}$ inches wide and 4 inches deep is supported at the ends and loaded at the centre so as to produce a deflection of $\frac{3}{4}$ inch, find the greatest inch stress on the fibres, taking $E = 25000000$ lbs. per square inch. Also find the load.*

Ans. Stress = 86805 lbs. per square inch;
Load = 19290 lbs.

(10) *For the same beam, if the greatest fibre stress is 12000 per square inch, find the greatest deflection.*

Ans. Deflection = 0.103 inches.

(11) *A rectangular oak beam* 1 *foot deep and* ½ *foot wide and* 15 *ft. long is fixed horizontally at one end and is free at the other end. Let the weight of the beam be* 54 *pounds per cubic foot. Suppose it sustains a uniform load of* 100 *pounds per foot extending over* 4 *feet of the beam, beginning at* 5 *feet from the fixed end. Also a weight of* 100 *pounds placed at* 11 *feet from the fixed end. Let* $E = 2000000$ *lbs. per square inch. Find the deflection at the free end.*

Ans. Deflection due to weight of beam = 0.17086 inch;
" " " uniform load = 0.12627 "
" " " the weight = 0.0684 "
Total deflection = 0.36553 inch.

(12) *If the same beam is loaded with five equal weights of* 100 *lbs. each at intervals of* 3 *feet, what is the deflection at the free end and at the third loaded point from the fixed end?*

Ans. Total deflection at free end = 0.27 inch.
" " " third point = 0.12555 inch.

(13) *Same beam supported at the ends. Find the central deflection due to its own weight.*

Ans. Deflection = 0.001483 ft.

(14) *A beam of pine weighing* 40 *lbs. per cubic foot,* 18¾ *inches deep,* 15 *inches wide,* 12½ *ft. long, is supported at the ends and has a weight of* 17935 *lbs. placed at* 48 *inches from one end. Find the deflection at centre and point of application of the weight when* $E = 1680000$ *lbs. per square inch.*

Ans. Deflection at centre due to weight of beam = 0.0032 inch.
" " " " " weight added = 0.078617 "
" " 48 in. " " weight of beam = 0.0027 "
" " 48 " " " weight added = 0.07185 "

(15) *A wrought-iron* 15-*inch I beam, whose moment of inertia is* 691 *in inches, has a length of* 30 *feet.* $E = 24000000$ *lbs. per square inch. If supported at the ends and a uniform load of* 75 *lbs. per inch of length covers the first* 10 *feet, find the deflection at the end of the load.*

Ans. Deflection = 0.23444 inch.

Find the deflection at the centre of the beam.

Ans. Deflection = 0.24421 inch.

Find the deflection 10 *feet from the unloaded end.*

Ans. Deflection = 0.19537 inch.

Where is the point of greatest deflection and what is the greatest deflection?

Ans. At 13.1676 feet. Greatest deflection = 0.24847 inch.

If the weight of the beam itself is 5.573 *lbs. per inch of length, find the deflection at the centre.*

Ans. Deflection = 0.07349 inch.

If the same 10-*foot load is moved along to the centre, find the deflection at the centre.*

Ans. Deflection = 0.50063 inch.

If the uniform load of 75 *lbs. per inch covers the whole span, what is the central deflection?*

Ans. Deflection = 0.98905 inch.

If the same beam is half loaded with 75 *pounds per inch, what is the deflection at the centre? What is the maximum deflection? and at what point is it?*

Ans. Deflection = 0.494525 inch. Max. deflection = 0.49855 inch.
Within the loaded portion at 14.48 inches from centre.

If the same beam has three weights of 4500 *lbs. each, placed at intervals of* 60 *inches beginning at one end, what is the deflection at the centre?*

Ans. Deflection = 0.6154 inch.

If there are eight weights each equal to 3000 *lbs. at intervals of* 40 *inches, what is the central deflection?*

Ans. Deflection = 0.97926 inch.

(16) *Suppose the same beam as in* (15) *to be fixed horizontally at both ends and loaded uniformly with* 75 *lbs. per inch. What is the deflection at* 10 *feet from either end? At the centre?*

Ans. Deflection = 0.1563 inch; at centre = 0.19781 inch.

(17) *If only one end is fixed, the other supported, what is the deflection at* 10 *feet? at centre? at* 20 *feet? What is the maximum deflection? Where is it?*

Ans. Deflection at 10 feet = 0.39074 inch; at centre = 0.39563 inch; at 20 feet = 0.27352 inch.
Maximum deflection = 0.41018 "
At 151.7524 inches from supported end.

(18) *Same beam as* (15) *fixed horizontally at both ends, with a concentrated load of* 27000 *lbs. If the load is at the centre, what is the deflection at half way between the centre and either end? What is central deflection? Where are the points of inflection?*

Ans. Deflection = 0.19781 inch; central deflection = 03.9562 inch.
At 90 inches from each end.

If the load is 7.5 *feet from the left end, where and what is the maximum deflection?*

Ans. Maximum deflection = 0.2136 inch; at 12 feet from left end.

If only the right end is fixed and the other supported, and the load of 27000 *lbs. is at the centre, what are the deflections at the quarter points? The centre? What is the maximum deflection?*

Ans. At the quarter points deflection = 0.5316, 0.3091 inch.
Central deflection = 0.69234 inch; maximum deflection = 0.70732 inch.

At $l\sqrt{\frac{1}{5}}$ from supported ends.

(19) *Same beam as* (15) *fixed horizontally at both ends has three weights of* 4500 *lbs. each placed at intervals of* 60 *inches, beginning at the left end. Find the central deflection.*

Ans. Deflection = 0.13187 inch.

If two other equal weights of 4500 *lbs. are added at the same interval of* 60 *inches, find the central deflection due to these last two weights.*

Ans. Deflection = 0.06594 inch.

Suppose the fifth weight removed, what is the deflection at the fourth weight? at the third and second weights?

Ans. Fourth-weight deflection = 0.13748 inch;
Third- " " = 0.18072 "
Second- " " = 0 1458 "

What are the end moments due to these four weights? and where are the points of contrary flexure?

Ans. $M_1 = +750000$ inch-pounds; $M_2 = -600000$ inch-pounds; 74.806 and 275.294 inches.

(20) *Let the ratio $\frac{l}{\kappa}$ of the length l of a strut to the least radius of gyration κ of its cross-section A be $\frac{l}{\kappa} = 100$. Let the cross-section be symmetrical. If the elastic limit is $S_e = 30000$ lbs. per square inch and the coefficient of elasticity is $E = 27000000$ lbs. per square inch, find the crippling unit load $\frac{W}{A}$ for two pin ends, for one pin and one flat end and for two flat ends.*

Ans. The limiting ratio $\frac{L}{\kappa}$ is 170, 190, 212 for two pin ends, one pin and one flat end, and two flat ends respectively. We therefore use either Gordon's formula or the formula (SP).

By Gordon's formula we have, since $v = v_1$,

$$\frac{W}{A} = \frac{30000}{1 + \frac{10000}{1800n^2\pi^2}};$$

and substituting the value of $n^2\pi^2$, we have

$$\frac{W}{A} = 22270,\ 23490,\ 24550 \text{ lbs. per square inch}$$

for two pin ends, one pin and one flat end, and two flat ends respectively.

By the formula (SP) we have

$$\frac{W}{A} = 30000\left[1 - \frac{100000}{3600n^2\pi^2}\right].$$

Hence

$$\frac{W}{A} = 24810,\ 25860,\ 26670 \text{ lbs. per square inch}$$

for two pin ends, one pin and one flat end, and two flat ends respectively.

We must divide the crippling load by the assumed factor of safety (page 291) for the working load. Thus if the factor of safety is taken at 4, we have from Gordon's formula 5567, 5870, 6137 lbs. per square inch, or from formula (SP) 6200, 6465, 6667 lbs. per square inch.

Again, from page 292, we obtain for repeated stress, if there is no steady stress, $S_w = 7500$, and putting this for S_e in formula (SP) and in the numerator of Gordon's formula, we obtain the same results as before for a factor of safety of 4.

If the steady stress is not zero but equal to the total stress, we have $S_w = 15000$, and using this for S_e we get the same results as if we had taken a factor of safety of 2.

For other ratios of steady to total stress we get the same results as if we had taken a factor of safety between 2 and 4.

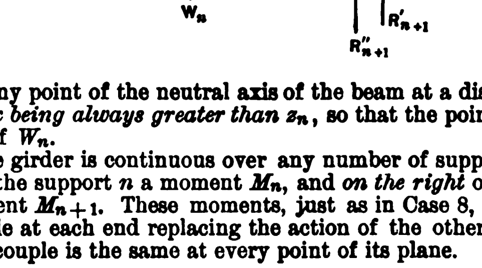
CHAPTER IV.

APPLICATIONS OF STATICS—THEORY OF FLEXURE—CONTINUOUS GIRDER.

CONTINUOUS GIRDER — CONDITIONS OF EQUILIBRIUM. EQUATION OF THE CURVE OF DEFLECTION. THEOREM OF THREE MOMENTS. DETERMINATION OF THE MOMENT AT ANY SUPPORT. RECAPITULATION—GENERAL FORMULAS.

Continuous Girder.—A beam or girder which rest upon more than two supports is called a continuous beam or girder. When a beam rests upon two supports only, a weight placed anywhere upon it causes pressures or reactions at the two supports which may be at once determined by the law of the lever. That is, the reactions are inversely as the segments of the span or either side of the weight. But when the beam is continuous over more than two supports this law no longer holds.

Conditions of Equilibrium.—Let l_n be the length of the nth span of a continuous beam, counting from the left-end support, so that n is the number of the support on the left and $n+1$ is the number of the support on the right. Take a point o vertically above the nth support as origin, and the horizontal through o as the axis of abscissas. Let there be a load W_n in this span l_n at a distance z_n from the left end. Let the reaction at the left end or nth support due to this load be R'_n, and at the right end or $n+1$th support R''_{n+1}.



Let P be any point of the neutral axis of the beam at a distance x from the left end, *x being always greater than z_n*, so that the point P is always *on the right* of W_n.

Now if the girder is continuous over any number of supports, we have *on the left* of the support n a moment M_n, and *on the right* of the support $n+1$ a moment M_{n+1}. These moments, just as in Case 8, page 842, are due to a couple at each end replacing the action of the other spans. The moment of a couple is the same at every point of its plane.

The necessary conditions of equilibrium for the span l_n are then:

1st. The algebraic sum of all the horizontal forces must be zero. There are in this case no horizontal forces and therefore this condition is fulfilled.

2d. The algebraic sum of all the vertical forces must be zero. We have therefore

$$R'_n + R''_{n+1} = W_n. \qquad (1)$$

3d. The algebraic sum of the moments of all the forces about any point P must be zero. Denoting by M_n the moment *on the left* of the support n, and by M_x the moment *on the left* of any point P, we have

$$M_n - R'_n x + W_n(x - z_n) - M_x = 0,$$

or

$$M_x = + M_n - R'_n x + W_n(x - z_n). \qquad (2)$$

If in this equation we make $x = l_n$, M_x becomes the moment M_{n+1} *on the left* of the support $n + 1$, and we have

$$M_{n+1} = + M_n - R'_n l_n + W_n(l_n - z_n). \qquad (3)$$

If we put the ratio $\frac{z_n}{l_n} = a_n$, we obtain from (3) for the reaction R'_n at the left support due to W_n, in terms of the moments M_n and M_{n+1} *on the left* of supports n and $n + 1$,

$$R'_n = \frac{M_n - M_{n+1}}{l_n} + W_n(1 - a_n). \qquad (4)$$

From this equation and (1) we have for the reaction R''_{n+1} at the support $n + 1$ due to W_n

$$R''_{n+1} = \frac{M_{n+1} - M_n}{l_n} + W_n a_n. \qquad (5)$$

The total reaction R_n at any support n is evidently equal to the sum of the reactions R'_n and R''_n just on the right and left.

We have from (5), for a load W_{n-1} in the preceding span l_{n-1},

$$R''_n = \frac{M_n - M_{n-1}}{l_{n-1}} + W_{n-1}a_{n-1}, \qquad (6)$$

where M_n and M_{n-1} are the moments *on the left* of the supports $n - 1$ and n.

The total reaction at the nth support is then

$$R_n = R''_n + R'_n. \qquad (7)$$

If there are any number of concentrated loads, we have only to put

$$\sum_n^{n+1} W_n(1 - a_n) \quad \text{and} \quad \sum_{n-1}^{n} W_{n-1}a_{n-1},$$

in place of $W_n(1 - a_n)$ and $W_{n-1}a_{n-1}$ in (4) and (6).

If, instead of concentrated loads, we have a uniform load w_{n-1} per unit of length over the span l_{n-1} and w_n per unit of length over the span l_n, we have $w_{n-1}dz_{n-1}$, or $w_{n-1}l_{n-1}da$ in place of W_{n-1} and $w_n dz_n$, or $w_n l_n da$ in place of W_n. If we make this substitution, we have

$$\int_0^1 w_{n-1}l_{n-1}a\,da = \frac{1}{2}w_{n-1}l_{n-1} \quad \text{and} \quad \int_0^1 w_n l_n(1 - a)da = \frac{1}{2}w_n l_n,$$

in place of $W_{n-1}a_{n-1}$ and $W_n(1 - a_n)$.

We have then in all cases, in general, for the reactions R'_n and R''_n, right and left of any support n,

$$\left.\begin{aligned} R'_n &= \frac{M_n - M_{n+1}}{l_n} + q'_n; \\ R''_n &= \frac{M_n - M_{n-1}}{l_{n-1}} + q''_{n-1}; \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (I)$$

where M_{n-1}, M_n and M_{n+1} are the moments *on the left* of supports $n - 1$, n and $n + 1$.

For concentrated loads

$$q'_n = \sum_n^{n+1} W_n(1 - a_n), \qquad q''_{n-1} = \sum_{n-1}^{n} W_{n-1}a_{n-1},$$

and *for uniform loading*

$$q'_n = \frac{1}{2} w_n l_n, \qquad q''_{n-1} = \frac{1}{2} w_{n-1} l_{n-1}.$$

From equations (I) we can then find in any case the reactions R''_n, R'_n just to left and right of any support n, provided we know the moments *on the left* of supports $n - 1$, n and $n + 1$. Counter-clockwise moments are positive and upward reactions are positive. If there is no load in the span l_n, q_n is zero. If there is no load in the span l_{n-1}, q_{n-1} is zero.

Equation of the Curve of Deflection.—We can now easily deduce the equation of the curve of deflection for a continuous beam for *constant moment of inertia* of cross-section I.

The differential equation of the curve of deflection is (page 326), taking moments on the left of any point,

$$EI\frac{d^2y}{dx^2} = - M_x,$$

where E is the coefficient of elasticity, I is the constant moment of inertia of the cross-section, and we take the minus sign for moments on the left of the point P.

Inserting the value of M_x from (2), we have

$$\frac{d^2y}{dx^2} = - \frac{M_n - R'_n x + W_n(x - z_n)}{EI}.$$

We can integrate this expression between the limits $x = 0$ and x, upon the condition that x is always greater than z_n, that is, *the point considered always on the right of the weight.* When, therefore, $x = 0$, $(x - z_n)$ must be zero. We must therefore take the integral of $W_n(x - z_n)$ simultaneously between the limits $x = z_n$ and x, or treat $(x - z_n)$ as a variable which becomes zero when $x = 0$.

We have then, integrating once,

$$\frac{dy}{dx} = - \frac{2M_n x - R'_n x^2 + W_n(x - z_n)^2}{2EI} + C,$$

where for $x = 0$ the constant of integration $C = \frac{dy}{dx}$ for $x = 0$, or equals the tangent t_n of the angle which the tangent at the support n to the curve makes with the horizontal. Hence

$$\frac{dy}{dx} = t_n - \frac{2M_n x - R'_n x^2 + W_n(x - z_n)^2}{2EI}. \quad \ldots \ldots \quad (3)$$

If we take the origin at a distance h_n above the support n (see figure page 367) and integrate again, the constant of integration for $x = 0$ will be $-h_n$, and we have

$$y = -h_n + t_n x - \frac{3M_n x^2 - R'_n x^3 + W_n(x - z_n)^3}{6EI}, \quad . \quad . \quad . \quad (9)$$

which is the general equation of the curve of deflection.

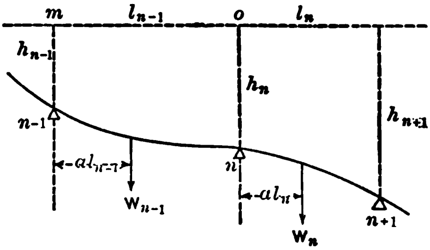
If in this we make $x = l_n$, y becomes $-h_{n+1}$. If we also put the ratio $\frac{z_n}{l_n}$ of the distance of the weight from the left end of span to the length of span, equal to a_n, so that $\frac{z_n}{l_n} = a_n$, and insert for R'_n its value as given by (4), we have from (9)

$$t_n = -\frac{h_{n+1} - h_n}{l_n} + \frac{1}{6EI}[2M_n l_n + M_{n+1} l_n - W_n l_n^2(2a_n - 3a_n^2 + a_n^3)]. [10]$$

We see, therefore, that the equation of the curve of deflection (9) is completely determined when we know M_n and M_{n+1}, the moments *at the left* of the two supports of the loaded span.

Theorem of Three Moments.—These moments are readily found by the application of the "theorem of three moments" which we shall now deduce.

Consider two consecutive spans l_{n-1} and l_n over the consecutive supports $n - 1$, n and $n + 1$. The equation of the curve of deflection between W_n



and the $n + 1$th support is given by (9), and the tangent of the angle which the curve makes with the horizontal is given by (8).

If in (8) we substitute for R'_n its value as given by (4), and for t_n its value from (10), and make at the same time $x = l_n$, then $\frac{dy}{dx}$ in (8) becomes t_{n+1} or the tangent at the $n + 1$th support, and we have

$$t_{n+1} = -\frac{h_{n+1} - h_n}{l_n} - \frac{1}{6EI}[M_n l_n + 2M_{n+1} l_n - W_n l_n^2(a_n - a_n^3)]. \quad [11]$$

Equation (11) gives the tangent of the angle which the tangent to the curve of deflection at the $n + 1$th support makes with the horizontal.

If we suppose a load W_{n-1} in the span l_{n-1} at a distance $a_{n-1} l_{n-1}$ from the left end, the origin being taken at m instead of at o, we can find from (11) the tangent t_n at the right end by diminishing each of the subscripts by unity. Hence we can write at once, from (11),

$$t_n = -\frac{h_n - h_{n-1}}{l_{n-1}} - \frac{1}{6EI} M_{n-1} l_{n-1} + 2M_n l_{n-1} - W_{n-1} l^2_{n-1}(a_{n-1} - a_{n-1}^3)]. (12)$$

But equation (10) gives us t_n for a load W_n in the span l_n. Let both W_{n-1} and W_n act, then, and since there is a common tangent at n for the curve on each side of support n, we have, by equating (10) and (12),

$$M_{n-1}l_{n-1}+2M_n(l_{n-1}+l_n)+M_{n+1}l_n=6EI\left[\frac{h_{n-1}-h_n}{l_{n-1}}+\frac{h_{n+1}-h_n}{l_n}\right]$$
$$+W_nl_n^2(2a_n-3a_n^2+a_n^3)+W_{n-1}l^2_{n-1}(a_{n-1}-a^3_{n-1}). \quad . \quad . \quad (13)$$

If there are any number of concentrated loads in each span l_{n-1} and l_n, we have only to put

$$\sum_n^{n+1} W_nl_n^2(2a_n-3a_n^2+a_n^3) \quad \text{and} \quad \sum_{n-1}^{n} W_{n-1}l^2_{n-1}(a_{n-1}-a^3_{n-1})$$

in place of the two last terms.

If, instead of concentrated loads, we have a uniform load w_{n-1} per unit of length over the span l_{n-1} and w_n per unit of length over the span l_n, we have $w_{n-1}dz_{n-1}$ in place of W_{n-1}, and w_ndz_n in place of W_n. Since the ratio $\frac{z_n}{l_n}$ or $\frac{z_{n-1}}{l_{n-1}}$ is denoted by a, we have $al_{n-1}=z_{n-1}$, and $al_n=z_n$. We can then put $w_{n-1}l_{n-1}da$ in place of W_{n-1}, and w_nl_nda in place of W_n. If we make this substitution, we have

$$\int_{a=0}^{a=1} w_{n-1}l^3_{n-1}(a-a^3)da=\frac{1}{4}w_{n-1}l^3_{n-1},$$

$$\int_{a=0}^{a=1} w_nl_n^3(2a-3a^2+a^3)da=\frac{1}{4}w_nl_n^3.$$

We have then in general

$$M_{n-1}l_{n-1}+2M_n(l_{n-1}+l_n)+M_{n+1}l_n=Y_n+A_n+B_{n-1}, \quad . \quad \text{(II)}$$

where we have for the sake of convenience of notation

$$Y_n=6EI\left[\frac{h_{n-1}-h_n}{l_{n-1}}+\frac{h_{n+1}-h_n}{l_n}\right];$$

for concentrated loads,

$$A_n=\sum_n^{n+1} W_nl_n^2(2a_n-3a_n^2+a_n^3), \qquad B_{n-1}=\sum_{n-1}^{n} W_{n-1}l^2_{n-1}(a_{n-1}-a^3_{n-1});$$

for uniform loading,

$$A_n=\frac{1}{4}w_nl_n^3, \quad B_{n-1}=\frac{1}{4}w_{n-1}l^3_{n-1}.$$

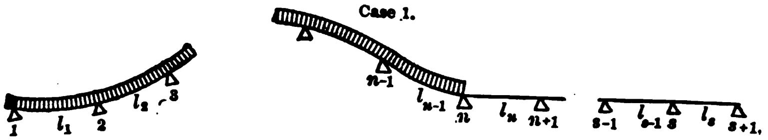
Equation (II) is the general form of the "theorem of three moments" for constant moment of inertia of cross-section. It gives the relation between the moments *at the left of* any three consecutive supports, $n-1$, n and $n+1$ of a continuous girder in terms of the consecutive spans l_{n-1} and l_n, the loading in those spans and the relative heights of the supports, provided the moment of inertia of the cross-section is constant.

If the supports are all on the same level, the term Y_n is zero and disappears. If there is no loading in the span l_n, the term A_n is zero and disappears. If there is no loading in the span l_{n-1}, the term B_{n-1} is zero and disappears.

Determination of the Moment M_n at Any Support. — Let us number the supports 1, 2, 3, etc., beginning at the left. The corresponding spans are l_1, l_2, l_3, etc. Let the entire number of spans be s. Then

the last span is l_s, and the last support is $s + 1$. If the extreme ends are not fixed, but simply rest upon the end supports, the moments M_1 and M_{s+1} at the first and last support are zero.

Case 1. Let us take any number of spans s, and let all the spans on the left of the nth support be loaded in any manner, and all the left supports



be at different levels, while all spans on the right of the nth support are on level. Let the ends rest on the supports, so that $M_1 = 0$ and $M_{s+1} = 0$. Let in general

$$Y''_n = 6EI\left[\frac{h_{n-1} - h_n}{l_{n-1}}\right], \qquad Y'_n = 6EI\left[\frac{h_{n+1} - h_n}{l_n}\right],$$

so that

$$Y_n = Y''_n + Y'_n.$$

In the present case $Y'_n = 0$, since supports n and $n + 1$ are on the same level. We have then by the successive application of the theorem of three moments the following equations, since M_1 and M_{s+1} are zero:

$$\left.\begin{array}{ll}
(c_2) & 2M_2(l_1 + l_2) + M_3 l_2 = Y_2 + A_2 + B_1; \\
(c_3) & M_2 l_2 + 2M_3(l_2 + l_3) + M_4 l_3 = Y_3 + A_3 + B_2; \\
(c_4) & M_3 l_3 + 2M_4(l_3 + l_4) + M_5 l_4 = Y_4 + A_4 + B_3; \\
 & \qquad \text{etc.}; \\
(c_{n-1}) & M_{n-2}l_{n-2} + 2M_{n-1}(l_{n-2} + l_{n-1}) + M_n l_{n-1} \\
 & \qquad = Y_{n-1} + A_{n-1} + B_{n-2}; \\
(c_n) & M_{n-1}l_{n-1} + 2M_n(l_{n-1} + l_n) + M_{n+1}l_n = Y'_n + B_{n-1}; \\
(c_{n+1}) & M_n l_n + 2M_{n+1}(l_n + l_{n+1}) + M_{n+2}l_{n+1} = 0; \\
 & \qquad \text{etc.}; \\
(c_{s-2}) & M_{s-3}l_{s-3} + 2M_{s-2}(l_{s-3} + l_{s-2}) + M_{s-1}l_{s-2} = 0; \\
(c_{s-1}) & M_{s-2}l_{s-2} + 2M_{s-1}(l_{s-2} + l_{s-1}) + M_s l_{s-1} = 0; \\
(c_s) & M_{s-1}l_{s-1} + 2M_s(l_{s-1} + l_s) = 0.
\end{array}\right\} \quad (15)$$

The solution of these equations (15) can be best effected by the method of indeterminate coefficients. Thus we multiply the first equation by a number c_2, the second by a number c_3, etc., the subscript corresponding always to that of M in the middle term. Having performed these multiplications, add the resulting equations and arrange the terms according to the coefficients of M_2, M_3, etc. We thus obtain the equation

$$\begin{aligned}
&[2c_2(l_1 + l_2) + c_3 l_2]M_2 + [c_2 l_2 + 2c_3(l_2 + l_3) + c_4 l_3]M_3 + \text{etc.}; \\
&\qquad + [c_{n-1}l_{n-1} + c_n(l_{n-1} + l_n) + c_{n+1}l_n]M_n + \text{etc.}; \\
&\qquad + [c_{s-1}l_{s-1} + 2c_s(l_{s-1} + l_s)]M_s \\
&= (Y'_n + B_{n-1})c_n + \sum_{n-1}' (Y_n + A_n + B_{n-1})c_n. \quad . \quad . \quad . \quad (16)
\end{aligned}$$

In order, then, to determine M_s we have only to impose such conditions upon the multipliers c that all terms on the left except the last in equation (16) shall be zero. We have then, assuming $c_1 = 0$ and $c_2 = 1$,

$$c_3 = -2\frac{l_1 + l_2}{l_2}, \quad c_4 = -2c_3\frac{l_2 + l_3}{l_3} - c_2\frac{l_2}{l_3}, \quad c_5 = -2c_4\frac{l_3 + l_4}{l_4} - c_3\frac{l_3}{l_4},$$

and generally for any multiplier c

$$c_n = -2c_{n-1}\frac{l_{n-2} + l_{n-1}}{l_{n-1}} - c_{n-2}\frac{l_{n-2}}{l_{n-1}}. \quad . \quad . \quad . \quad . \quad (17)$$

These values of c make all terms zero on the left of equation (16) except the last, and give us for the value of M_s

$$M_s = \frac{(Y''_n + B_{n-1})c_n + \sum\limits_{n-1}'(Y_n + A_n + B_{n-1})c_n}{c_{s-1}l_{s-1} + 2c_s(l_{s-1} + l_s)}. \quad . \quad . \quad (18)$$

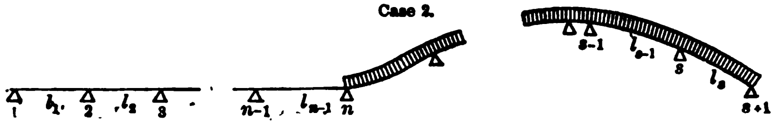
From the law of the multipliers we have

$$c_{s-1}l_{s-1} + 2c_s(l_{s-1} + l_s) + c_{s+1}l_s = 0.$$

Hence we may put in the denominator of (18) the equivalent expression $-c_{s+1}l_s$.

Case 2. Let all the spans on the right of the nth support be loaded in any manner and all the right supports be at different levels, while all the spans on the left of the nth support are unloaded and all the left supports are on level. As before $M_1 = 0$ and $M_{s+1} = 0$.

Case 2.



In the present case $Y''_n = 0$, since supports n and $n - 1$ are on level. We have then by successive applications of the theorem of three moments the following equations:

$$\left.\begin{array}{ll}
(d_s) & 2M_2(l_1 + l_2) + M_3l_2 = 0; \\
(d_{s-1}) & M_2l_2 + 2M_3(l_2 + l_3) + M_4l_3 = 0; \\
(d_{s-2}) & M_3l_3 + 2M_4(l_3 + l_4) + M_5l_4 = 0; \\
 & \qquad \text{etc.}; \\
(d_{s-n+3}) & M_{n-2}l_{n-2} + 2M_{n-1}(l_{n-2} + l_{n-1}) + M_nl_{n-1} = 0; \\
(d_{s-n+2}) & M_{n-1}l_{n-1} + 2M_n(l_{n-1} + l_n) + M_{n+1}l_n = Y_n + A_n; \\
(d_{s-n+1}) & M_nl_n + 2M_{n+1}(l_n + l_{n+1}) + M_{n+2}l_{n+1} \\
 & \qquad = Y_{n+1} + A_{n+1} + B_n; \\
 & \qquad \text{etc.}; \\
(d_3) & M_{s-3}l_{s-3} + 2M_{s-2}(l_{s-3} + l_{s-2}) + M_{s-1}l_{s-2} \\
 & \qquad = Y_{s-2} + A_{s-2} + B_{s-3}; \\
(d_2) & M_{s-2}l_{s-2} + 2M_{s-1}(l_{s-2} + l_{s-1}) + M_sl_{s-1} \\
 & \qquad = Y_{s-1} + A_{s-1} + B_{s-2}; \\
(d_1) & M_{s-1}l_{s-1} + 2M_s(l_{s-1} + l_s) = Y_s + A_s + B_{s-1}.
\end{array}\right\} \quad (19)$$

If we multiply the last of equations (19) by a number d_2, the last but one by d_3, the nth by d_{s-n+2}, etc., add the resulting equations and arrange the terms according to the coefficients of M_s, M_s, etc., we obtain

$$\begin{aligned}[2d_2(l_{s-1}+l_s) + d_3 l_{s-1}]M_s + [d_2 l_{s-1} + 2d_3(l_{s-2} + l_{s-1}) + d_4 l_{s-2}]M_{s-1} + \text{etc.};\\ + [d_{s-n+1} l_n + 2(l_{n-1} + l_n)d_{s-n+2} + d_{s-n+3} l_{n-1}]M_n + \text{etc.};\\ + [d_{s-2} l_3 + 2d_{s-1}(l_3 + l_2) + d_s l_2]M_3 + [d_{s-1} l_2 + 2d_s(l_1 + l_2)]M_2\\ = (Y'_n + A_n)d_{s-n+2} + \sum_{n+1}^{n+1}(Y_n + A_n + B_{n-1})d_{s-n+2}. \quad (20)\end{aligned}$$

In order to determine M_2 we have only to impose such conditions upon the multipliers d that all terms on the left except the last in equation (20) shall be zero. We have then, assuming $d_1 = 0$, $d_2 = 1$,

$$d_3 = -2\frac{l_s + l_{s-1}}{l_{s-1}}, \quad d_4 = -2d_3\frac{l_{s-1} + l_{s-2}}{l_{s-2}} - d_2\frac{l_{s-1}}{l_{s-2}},$$

and generally for any multiplier d,

$$d_n = -2d_{n-1}\frac{l_{s-n+3} + l_{s-n+2}}{l_{s-n+2}} - d_{n-2}\frac{l_{s-n+3}}{l_{s-n+2}}. \quad (21)$$

These values of d make all terms zero on the left of equation (20) except the last, and give us for the value of M_2

$$M_2 = \frac{(Y'_n+A_n)d_{s-n+2} + \sum_{s+1}^{n+1}(Y_n + A_n + B_{n-1})d_{s-n+2}}{d_{s-1}l_2 + 2d_s(l_2 + l_1)}. \quad (22)$$

From the law of the multipliers we have

$$d_{s-1}l_2 + 2d_s(l_2 + l_1) + d_{s+1}l_1 = 0.$$

Hence we may put in the denominator of (22) the equivalent expression $-d_{s+1}l_1$.

Now from equations (19) and from the values of c given by (17) we see at once by inspection that

$$M_3 = c_3 M_2, \; M_4 = c_4 M_2, \text{ etc., and generally } M_m = c_m M_2,$$

and this holds good *so long as m is less than n.*

We have then for the moment M_m at any support m on the left of the nth in the second case,

for $m < n$

$$M_m = \frac{c_m d_{s-n+2}(Y'_n + A_n) + c_m\sum_{s+1}^{n+1}(Y_n + A_n + B_{n-1})d_{s-n+2}}{d_{s-1}l_2 + 2d_s(l_1 + l_2) \text{ or } -d_{s+1}l_1}. \quad (23)$$

Again, from equations (15) and from the values of d given by (21) we see at once by inspection that

$$M_{s-1} = d_3 M_s, \quad M_{s-2} = d_4 M_s, \text{ etc., and generally } M_m = d_{s-m+2}M_s,$$

and this holds good so long as m is greater than n.

We have then for the moment M_m at any support m on the right of the nth in the first case,

for $m > n$

$$M_m = \frac{c_n d_{s-m+2}(Y''_n + B_{n-1}) + d_{s-m+2}\sum_{n-1}(Y_n + A_n + B_{n-1})c_n}{c_{s-1}l_{s-1} + 2c_s(l_{s-1} + l_s), \text{ or } -c_{s+1}l_s}. \quad (24)$$

If we make in (21) and (17) $n = s + 1$ and then give different values to s and compare the results, we see that in general $c_{s+1}l_s = d_{s+1}l_1$. The denominators in (22) and (23) are then the same.

If we suppose Case 1 and Case 2 to exist simultaneously, we have the case of all spans loaded and all supports on different level. If then we make $m = n$ in (23) and (24) and add these two equations, we have, since $Y'_n + Y''_n = Y_n$, for the moment M_n on the left of any support n

$$M_n = \frac{d_{s-n+2}\sum\limits_n (Y_n + A_n + B_{n-1})c_n + c_n \sum\limits_{s+1}^{n+1} (Y_n + A_n + B_{n-1})d_{s-n+2}}{D}, \quad \text{(III)}$$

where we can put for the denominator D any one of the equivalent values

$$D = c_{s-1}l_{s-1} + 2c_s(l_{s-1} + l_s) = -c_{s+1}l_s = -d_{s+1}l_1 = d_{s-1}l_2 + 2d_s(l_1 + l_2).$$

Equation (III) gives the moment with its proper sign *on the left of any support n*. If we wish the moment on the right of any support n, we must change the sign for M_n as given by (III).

Recapitulation—General Formulas.—We have then for the moment *on the left* of any support n of a continuous girder of constant moment of inertia of cross-section, for any loading and any levels of supports,

$$M_n = \frac{d_{s-n+2}\sum\limits_n (Y_n + A_n + B_{n-1})c_n + c_n \sum\limits_{s+1}^{n+1} (Y_n + A_n + B_{n-1})d_{s-n+2}}{D}, \quad \text{(III)}$$

where we can put for D any one of the equivalent values

$$D = c_{s-1}l_{s-1} + 2c_s(l_{s-1} + l_s) = -c_{s+1}l_s = -d_{s+1}l_1 = d_{s-1}l_2 + 2d_s(l_1 + l_2). \quad (1)$$

In this equation s is the number of spans,

$$Y_n = 6EI\left[\frac{h_{n-1} - h_n}{l_{n-1}} + \frac{h_{n+1} - h_n}{l_n}\right], \quad . \; . \; . \; . \; . \; . \quad (2)$$

where h_{n-1}, h_n and h_{n+1} are the distances below any assumed level line of the three consecutive supports $n - 1$, n and $n + 1$.

For concentrated loads

$$A_n = \sum_n^{n+1} W_n l_n^2(2a_n - 3a_n^2 + a_n^3), \quad B_{n-1} = \sum_{n-1}^{n} W_{n-1}l^2_{n-1}(a_{n-1} - a^3_{n-1}),$$

where W_n is a load in span l_n, and W_{n-1} a load in span l_{n-1}, and a is the ratio of the distance of any load from the left end of its span to the length of the span, or $a = \frac{z_n}{l_n}$.

For uniform loading

$$A_n = \frac{1}{4}w_n l_n^3, \quad B_{n-1} = \frac{1}{4}w_{n-1}l^3_{n-1},$$

where w_n and w_{n-1} are the loads per unit of length over spans l_n and l_{n-1}.

The numbers c are given by

$$c_1 = 0, \; c_2 = 1, \text{ and for any other } c_n = -2c_{n-1}\frac{l_{n-2} + l_{n-1}}{l_{n-1}} - c_{n-2}\frac{l_{n-2}}{l_{n-1}}. \quad (3)$$

The numbers d are given by

$$\left.\begin{array}{l} d_1 = 0, \quad d_2 = 1, \quad \text{and for any other} \\ d_n = -2d_{n-1}\dfrac{l_{s-n+3} + l_{s-n+2}}{l_{s-n+2}} - d_{n-2}\dfrac{l_{s-n+3}}{l_{s-n+2}} \end{array}\right\} \; . \; . \; . \; . \quad (4)$$

For the reaction just to the right of any support n we have

$$R'_n = \frac{M_n - M_{n+1}}{l_n} + q'_n,$$

and just to the left of any support n . . . . (I)

$$R''_n = \frac{M_n - M_{n-1}}{l_{n-1}} + q''_{n-1},$$

where M_{n-1}, M_n and M_{n+1} are the moments *on the left* of supports $n - 1$, n and $n + 1$.

For concentrated loads

$$q'_n = \sum_n^{n+1} W_n(1 - a_n), \quad q''_{n-1} = \sum_{n-1}^{n} W_{n-1}a_{n-1}, \quad . \; . \; . \; . \quad (5)$$

and *for uniform loading*

$$q'_n = \frac{1}{2} w_n l_n, \qquad q''_{n-1} = \frac{1}{2} w_{n-1} l_{n-1}. \quad . \; . \; . \; . \quad (6)$$

For the total reaction at any support

$$R_n = R'_n + R''_n. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

Moments counter-clockwise are positive and reactions upwards are positive. Equation (III) gives the moment with its proper sign *on the left of any support n*. If we wish the moment on the right, we must *change the sign* for M_n as given by (III).

Special Cases.—If the supports are all on level, equation (8) is zero and the Y's disappear in equation (1).

If the spans are all equal, we have

$$\left.\begin{array}{llll} c_1 = 0, & c_2 = 1, & c_3 = -4, & c_4 = +15, \text{ etc.}; \\ d_1 = 0, & d_2 = 1, & d_3 = -4, & d_4 = +15, \text{ etc.}; \end{array}\right\} \quad . \; . \; . \; . \quad (8)$$

or *the values of the c's and d's are the same. They are alternately + and −, and each one is numericallg equal to four times the preceding minus the one next preceding.*

If we make l_1 or $l_s = 0$, the beam is *fixed horizontally* at either the left or the right end. We must remember, however, that when we thus make l_1 or l_s equal to zero, the value of s must still remain unchanged and the supports must be numbered as they were before the end spans were made zero.

EXAMPLES.

(1) *A beam of one span of length l is fixed horizontally at the ends. Find the end moments and reactions for a load W at a distance $z = al$ from the left end. Also for a uniform load of w per unit of length over the span.*

Ans. Let there be three spans, l_1, l_2, l_3, and let l_1 and l_3 be zero. Then $s = 3$, and we have

$$c_1 = d_1 = 0, \quad c_2 = d_2 = 1, \quad c_3 = d_3 = -2.$$

We have also $Y_1 = Y_4 = 0$, $A_1 = A_3 = B_1 = B_3 = 0$. Hence for $n = 2$ we have in general from equation (III), page , for the moment M_2 *on the left* of the left end of the span

$$M_2 = \frac{d_3(Y_2 + A_2)c_2 + c_2(Y_3 + B_2)d_2}{ld_3 + 2ld_2} = \frac{2(Y_2 + A_2) - Y_3 - B_2}{3l}. \quad . \; . \quad (1)$$

For $n = 3$ we have for the moment M_3 *on the left* of the right end, from (III),

$$M_3 = \frac{d_2(Y_2 + A_2)c_2 + d_3(Y_3 + B_3)c_3}{ld_2 + 2ld_3} = -\frac{Y_2 + A_2 - 2(Y_3 + B_2)}{3l}. \quad . \quad (2)$$

***If the ends are on level,* $Y_2 = Y_3 = 0$, and**

$$M_2 = \frac{2A_2 - B_2}{3l}, \quad M_3 = -\frac{A_2 - 2B_2}{3l} \quad . \; . \; . \; . \; . \; . \; . \quad (3)$$

Inserting in (3) the values of A_2 and B_2 for concentrated load, we have *for concentrated load and ends level*

$$M_2 = + Wl(a - 2a^2 + a^3), \quad M_3 = Wl(a^2 - a^3).$$

These are precisely the same values, in different form, already found for the end moments in this case on page 343, except that M_3 is on the left instead of on the right.

For the reaction at the left end we have from (I), page 369,

$$R_2' = \frac{M_2 - M_3}{l} + W(1 - a) = + W(1 - 3a^2 + 2a^3),$$

and for the reaction at the right end

$$R_3'' = \frac{M_3 - M_2}{l} + Wa = + W(3a^2 - 2a^3).$$

These are precisely the same values, in different form, already found for the end reactions in this case, page 343.

***For uniform load and ends level* we have, inserting the values of A_2 and B_2 in (3),**

$$M_2 = + \frac{1}{12}wl^2, \quad M_3 = + \frac{1}{12}wl^2;$$

$$R_2' = + \frac{1}{2}wl, \quad R_3'' = + \frac{1}{2}wl.$$

These are the same values as obtained on page 345 for the case, except that M_3 is on the left instead of on the right.

For uniform load and ends out of level,

$$M_2 = \frac{2Y_2 - Y_3}{3l} + \frac{wl^2}{12}, \quad M_3 = -\frac{Y_2 - 2Y_3}{3l} + \frac{wl^2}{12}, \quad R_2' = \frac{Y_2 - Y_3}{l^2} + \frac{wl}{2}.$$

How much must the left end be lowered in order to make the left reaction R_2' equal to zero?

Here we have

$$\frac{Y_2 - Y_3}{l^2} + \frac{wl}{2} = 0, \quad \text{or} \quad Y_2 - Y_3 = -\frac{wl^3}{2}.$$

Since $Y_3 = -Y_2$, we have

$$Y_2 = 6EI\left[\frac{h_3 - h_2}{l}\right] = -\frac{wl^3}{4}.$$

Hence

$$h_3 - h_2 = -\frac{wl^4}{24EI}$$

Since E is always very large, we see that a very small lowering of the left support will make the left reaction zero. We have in this case

$$M_2 = -\frac{wl^2}{6}, \quad M_3 = +\frac{wl^2}{3}, \quad R_3'' = + wl.$$

How much must the left end be lowered in order to make $M_3 = 0$?

Here we have

$$\frac{2Y_2 - Y_3}{3l} + \frac{wl^2}{12} = 0, \quad \text{and} \quad Y_3 = -Y_2.$$

Hence

$$Y_2 = 6EI\left[\frac{h_3 - h_2}{l}\right] = -\frac{wl^3}{12} \quad \text{and} \quad h_3 - h_2 = -\frac{wl^4}{72EI},$$

$$M_2 = +\frac{wl^2}{6}, \quad R_2' = +\frac{wl}{8}, \quad R_2'' = +\frac{2wl}{8}.$$

(2) *A beam of one span of length l is fixed horizontally at the right end. Find the reactions and moment at the right end for a load W at a distance $z = al$ from the left end. Also for a uniform load of w per unit of length over the span.*

Ans. Let there be two spans l_1 and l_2, and let $l_2 = 0$. Then $s = 2$, and we have

$$c_1 = d_1 = 0, \quad c_2 = d_2 = 1, \quad d_3 = -2, \quad h_2 - h_3 = 0, \quad Y_3 = 0,$$

$$A_2 = B_2 = 0, \quad Y_2 = 6EI\left[\frac{h_1 - h_2}{l}\right].$$

Hence for $n = 2$ we have in general from equation (III), page 375, for the moment M_2 *on the left* of the right end,

$$M_2 = \frac{Y_2 + B_1}{2l}.$$

We also have $M_1 = 0$, $M_3 = 0$. Hence

$$R_1' = -\frac{Y_2 + B_1}{2l^2} + q_1', \quad R_2'' = \frac{Y_2 + B_1}{2l^2} + q_1.$$

If the ends are level, $Y_2 = 0$ and

$$M_2 = \frac{B_1}{2l}, \quad R_1' = -\frac{B_1}{2l^2} + q_1', \quad R_2'' = \frac{B_1}{2l^2} + q_1''.$$

For concentrated load, ends level, we have then

$$M_2 = \frac{Wl}{2}(a - a^3), \quad R_1' = \frac{W}{2}(2 - 3a + a^3), \quad R_2'' = \frac{W}{2}(3a - a^3).$$

For uniform load, ends level, we have

$$M_2 = +\frac{wl^2}{8}, \quad R_1' = +\frac{3}{8}wl, \quad R_2'' = +\frac{5}{8}wl$$

How much must the left end be lowered in order to make the left reaction R_1' zero?

Here we have

$$-\frac{Y_2 + B_1}{2l^2} + q_1' = 0, \quad \text{or} \quad Y_2 = 6EI\left[\frac{h_1 - h_2}{l}\right] = -B_1 + 2q_1'l^2.$$

Hence

$$h_2 - h_1 = \frac{B_1 l - 2q_1'l^2}{6EI}, \quad M_2 = q_1'l, \quad R_2'' = +q_1' + q_1''.$$

If the load is uniform,

$$q_1' = q_1'' = \frac{wl}{2}, \quad B_1 = \frac{wl^3}{4}, \quad h_2 - h_1 = -\frac{wl^4}{8EI}, \quad M_2 = +\frac{wl^2}{2}, \quad R_2'' = wl.$$

If the load is concentrated,

$$q_1' = W(1-a), \quad q_1'' = Wa, \quad B_1 = Wl^2(a-a^3),$$

$$h_2 - h_1 = -\frac{Wl^3(2-3a+a^3)}{6EI}, \quad M_2 = Wl(1-a), \quad R_2'' = W.$$

How much must the right end be lowered in order that the moment M_2 may be zero?

Here we have

$$\frac{Y_2+B_1}{2l} = 0, \quad \text{or} \quad Y_2 = 6EI\left[\frac{h_1-h_2}{l}\right] = -B_1.$$

Hence

$$_1 - h_2 = -\frac{B_1 l}{6EI}, \quad R_1' = q_1', \quad R_2'' = q_1''.$$

If the load is uniform,

$$h_1 - h_2 = -\frac{wl^4}{24EI}, \quad R_1' = R_2'' = \frac{wl}{2}.$$

If the load is concentrated,

$$h_1 - h_2 = -\frac{Wl^3(a-a^3)}{6EI}, \quad R_1' = W(1-a), \quad R_2'' = Wa.$$

(3) *Find the general formulas for a continuous beam of two spans.*

Ans. Here $s = 2$, and we have from (III), page 375,

$$M_1 = 0, \quad M_3 = 0, \quad M_2 = \frac{Y_2 + A_2 + B_1}{2(l_1+l_2)}, \quad R_1' = -\frac{M_2}{l_1} + q_1',$$

$$R_2'' = \frac{M_2}{l_1} + q_1'', \quad R_2' = \frac{M_2}{l_2} + q_2', \quad R_3'' = -\frac{M_2}{l_2} + q_2'',$$

$$Y_2 = 6EI\left[\frac{h_1-h_2}{l_1} + \frac{h_3-h_2}{l_2}\right].$$

For concentrated loading,

$$q_1' = \Sigma W_1(1-a_1), \quad q_1'' = \Sigma W_1 a_1, \quad q_2' = \Sigma W_2(1-a_2), \quad q_2'' = \Sigma W_2 a_2,$$

$$A_2 = \Sigma W_2 l_2^2(2a_2 - 3a_2^2 + a_2^3), \quad B_1 = \Sigma W_1 l_1^2(a_1 - a_1^3).$$

For uniform loading,

$$q_1' = q_1'' = \frac{1}{2}w_1 l_1, \quad q_2'' = q_2' = \frac{1}{2}w_2 l_2, \quad A_2 = \frac{1}{4}w_2 l_2^3, \quad B_1 = \frac{1}{4}w_1 l_1^3.$$

These formulas will solve any case of two spans.

(4) *A plate girder is continuous over three supports, $l_1 = 80$ ft., $l_2 = 50$ ft., the supports being all on level. The uniform load per coot in the first span is $w_1 = 3000$ lbs., in the second $w_2 = 350$ lbs. Find the moments and reactions.*

Ans. From the general formulas of Example (3), since all supports are on level, $Y_2 = 0$, and we have

$$M_1 = 0, \quad M_3 = 0, \quad M_2 = \frac{A_2 + B_1}{2(l_1+l_2)}.$$

In the present case $A_2 = \dfrac{w_2 l_2^3}{4}$, $B_1 = \dfrac{w_1 l_1^3}{4}$. Hence

$$M_2 = \frac{w_1 l_1^3 + w_2 l_2^3}{8(l_1+l_2)} = \frac{3000 \times 80^3 + 350 \times 50^3}{8(80+50)} = +\,1494921.875 \text{ ft.-lbs.}$$

We have therefore

$$R_1' = -\frac{M_2}{l_1} + \frac{w_1 l_1}{2} = -\frac{194921.875}{30} + \frac{3000 \times 30}{2} = +38502.6 \text{ lbs.};$$

$$R_3'' = -\frac{M_2}{l_2} + \frac{w_2 l_2}{2} = -\frac{194921.875}{50} + \frac{350 \times 50}{2} = +4851.5625 \text{ lbs.};$$

$$R_2'' = \frac{M_2}{l_1} + \frac{w_1 l_1}{2} = +51497.39 \text{ lbs.}; \quad R_2' = \frac{M_2}{l_2} + \frac{w_2 l_2}{2} = +12648.44 \text{ lbs.}$$

$$R_2 = R_2'' + R_2' = +64145.8 \text{ lbs.}$$

How far must the second support be lowered in order that the moment M_2 may be zero?

Since supports 1 and 3 remain on level, $h_1 - h_2 = h_3 - h_2$. We have then

$$Y_2 = 6EI\left[\frac{h_1 - h_2}{l_1} + \frac{h_1 - h_2}{l_2}\right],$$

and

$$6EI\left[\frac{h_1 - h_2}{l_1} + \frac{h_3 - h_2}{l_2}\right] + \frac{w_2 l_2^3}{4} + \frac{w_1 l_1^3}{4} = 0, \text{ or } h_1 - h_2 = -\frac{w_1 l_1^4 l_2 + w_2 l_2^4 l_1}{24EI(l_1 + l_2)}.$$

If we take $E = 24000000$ lbs. per square inch, and if $I = 53400$ for dimensions in inches, we have

$$h_1 - h_2 = -0.054 \text{ inch.}$$

Therefore a sinking of the second support of only about $\frac{5}{100}$ of an inch is sufficient to make M_2 zero.

How far must the second support be lowered in order that the reaction on the second support may be zero?

Here we have

$$R_2'' + R_2' = R_2 = 0, \quad \text{or} \quad \frac{M_2}{l_1} + \frac{w_1 l_1}{2} + \frac{M_2}{l_2} + \frac{w_2 l_2}{2} = 0,$$

or

$$M_2 = -\frac{w l_1 l_2 + w_2 l_1 l_2^2}{2(l_1 + l_2)} = -1007812.5 \text{ ft.-lbs.}$$

From the general value of M_2 in Example (3),

$$-w_1 l_1^2 l_2 -- w_1 l_1 l_2^2 = \frac{w_1 l_1^3}{4} + \frac{w_2 l_2^3}{4} + 6EI\left[\frac{h_1 - h_2}{l_1} + \frac{h_3 - h^2}{l_2}\right].$$

Hence, since $h_1 - h_2 = h_3 - h_2$, $E = 24000000$, $I = 53400$,

$$h_1 - h_2 = -\frac{w_1 l_1^4 l_2 + w_2 l_1 l_2^4 + 4w_1 l_1^3 l_2^2 + 4w_2 l_1^2 l_2^3}{24EI(l_1 + l_2)} = -0.73 \text{ inch.}$$

Therefore a sinking of the second support of only about seven tenths of an inch is sufficient to *convert the two spans into one long span.*

We see then that a continuous girder requires the supports to be invariable.

We find in the present case

$$R_1' = +78593.75 \text{ lbs.}, \quad R_3'' = +28906.25 \text{ lbs.},$$

$$R_2'' = +11406.25 \text{ lbs.}, \quad R_2' = -11406\ 25 \text{ lbs.},$$

$$R_2 = R_2'' + R_2' = 0.$$

If the spans l_1 and l_2 are equal and w_1 and w_2 are equal, we have at once

$h_1 - h_2 = -\frac{5wl^4}{24EI}$, or the deflection at the centre of a span whose length is $2l$, and $M_2 = -\frac{wl^2}{2}$ as should be.

(5) *If in the case of Example* (4) *we have a concentrated load* $W_1 =$ 90000 *lbs. in the first span at a distance* $\frac{1}{4}l_1$ *from the left end, and a concentrated load* $W_2 =$ 18000 *lbs. at a distance* $\frac{1}{2}l_2$, *find the moments and reactions.*

Ans. We have

$$a_1 = \frac{1}{4}, \quad a_2 = \frac{1}{2}, \quad A_2 = W_2 l_2^2(2a_2 - 3a_2^2 + a_2^3) = \frac{3}{8}W_2 l_2^2,$$

$$B_1 = W_1 l_1^2(a_1 - a_1^3) = \frac{15}{64}W_1 l_1^2.$$

Then, from the general formulas of Example (3), we have

$M_2 = +\ 224121.094$ ft.-lbs., $R_1' = +\ 60029.3$ lbs., $R_3'' = +\ 4517.58$ lbs., $R_2'' = +\ 29970.7$ lbs., $R_2' = +13482.42$ lbs., $R_2 = R_2'' + R_2' = +\ 43453.12$ lbs.

For the distance the second support must be lowered in order that M_2 may be zero we find

$$h_1 - h_2 = -\ 0.1511 \text{ inch.}$$

For the distance the second support must be lowered in order that R_3 may be zero we find

$$h_1 - h_2 = -\ 0.55 \text{ inch.}$$

(6) *Let a beam of two equal spans have a load* W_1 *in the first span and* W_2 *in the second span, each load being at the middle of its span. Let the second support be lowered by an amount* $h_1 - h_2 = -\frac{(W_1 + W_2)l^3}{48EI}$. *What are the moments, shears and reactions?*

Ans. $M_2 = (W_1 + W_2)\frac{l}{32}$, $R_1' = \frac{15W_1 - W_2}{32}$, $R_3'' = \frac{15W_2 - W_1}{32}$,

$$R_2'' = \frac{17W_1 + W_2}{32}, \quad R_2' = \frac{W_1 + 17W_2}{32}, \quad R_2 = \frac{18(W_1 + W_2)}{32}.$$

(7) *Let a beam of two spans* l_1 *and* l_2 *level supports have a load* W_1 *at a distance* al_1 *from the left end of the first span. Find the reactions when* $l_1 = l$, *and* $l_2 = nl$.

Ans. $R_1' = \frac{W_1}{2(1+n)}[2(1+n) - a(3+2n) + a^3]$,

$$R_2'' = \frac{W_1}{2(1+n)}[a(3+2n) - a^3], \quad R_2' = \frac{W_1}{2n(1+n)}(a - a^3),$$

$$R_3'' = -\frac{W_1}{2n(1+n)}(a - a^3), \quad R_2 = R_2'' + R_2' = \frac{W_1}{2n}[a(1+2n) - a^3].$$

If the spans are equal, $n = 1$ and

$$R_1' = \frac{W_1}{4}[4 - 5a + a^3], \quad R_2'' = \frac{W_1}{4}[5a - a^3], \quad R_2' = \frac{W_1}{4}(a - a^3),$$

$$R_3'' = -\frac{W_1}{4}(a - a^3), \quad R_2 = R_2'' + R_2' = \frac{W_1}{2}(3a - a^3).$$

(8) *Find the general formulas for a continuous beam of three spans.*

Ans. Here $s = 3$, and we have from (III), page 375,

$$M_1 = 0, \quad M_4 = 0, \quad M_2 = -\frac{d_3(Y_2 + A_2 + B_1) + Y_3 + A_3 + B_2}{d_4 l_1},$$

$$M_3 = -\frac{Y_2 + A_2 + B_1 + c_3(Y_3 + A_3 + B_2)}{d_4 l_1};$$

$$R_1' = -\frac{M_2}{l_1} + q_1', \quad R_2'' = \frac{M_2}{l_1} + q_1'', \quad R_2' = \frac{M_2 - M_3}{l_2} + q_2',$$

$$R_3'' = \frac{M_3 - M_2}{l_2} + q_2'', \quad R_3' = \frac{M_3}{l_3} + q_3', \quad R_4'' = -\frac{M_3}{l_3} + q_3'';$$

$$c_3 = -2\frac{l_1 + l_2}{l_2}, \quad d_3 = -2\frac{l_2 + l_3}{l_2}, \quad d_4 = +4\frac{(l_1 + l_2)(l_2 + l_3)}{l_1 l_2};$$

$$Y_2 = 6EI\left[\frac{h_1 - h_2}{l_1} + \frac{h_3 - h_2}{l_2}\right], \quad Y_3 = 6EI\left[\frac{h_2 - h_3}{l_2} + \frac{h_4 - h_3}{l_3}\right].$$

For concentrated loads,

$$q_1' = \Sigma W_1(1 - a_1), \quad q_1'' = \Sigma W_1 a_1, \quad q_2' = \Sigma W_2(1 - a_2), \quad q_2'' = \Sigma W_2 a_2,$$
$$q_3' = \Sigma W_3(1 - a_3), \quad q_3'' = \Sigma W_3 a_3.$$

For uniform loading

$$q_1' = q_1'' = \frac{1}{2}w_1 l_1, \quad q_2' = q_2'' = \frac{1}{2}w_2 l_2, \quad q_3' = q_3'' = \frac{1}{2}w_3 l_3.$$

These formulas will solve any case of three spans.

(9) *Let a beam of three spans, level supports, have a load W_1 at a distance al_1 from the left end of the first span. Find the reactions when $l_1 = l_3 = l$ and $l_2 = nl$.*

Ans. For convenience of notation let

$$H = 4 + 8n + 3n^2.$$

Then

$$R_1' = \frac{W_1}{H}[(1 - a)H - (a - a^3)(2 + 2n)], \quad R_2'' = \frac{W_1}{H}[Ha + (a - a^3)(2 + 2n)],$$

$$R_2' = \frac{W_1}{H}[(a - a^3)\left(3 + \frac{2}{n}\right), \quad R_3'' = -\frac{W_1}{H}\left[(a - a^3)\left(3 + \frac{2}{n}\right)\right],$$

$$R_3' = -\frac{W_1}{H}(a - a^3)n, \quad R_4'' = \frac{W_1}{H}(a - a^3)n,$$

$$R_2 = R_2'' + R_2' = \frac{W_1}{H}\left[Ha + (a - a^3)\left(5 + 2n + \frac{2}{n}\right)\right],$$

$$R_3 = R_3'' + R_3' = -\frac{W_1}{H}\left[(a - a^3)\left(3 + n + \frac{2}{n}\right)\right].$$

(10) *A continuous beam of four equal spans, level supports, has the second span from the left covered with a uniform load of w per unit of length. Find the moments and reactions.*

Ans. $M_1 = 0, \quad M_2 = +\frac{11}{224}wl^2, \quad M_3 = +\frac{12}{224}wl^2, \quad M_4 = -\frac{3}{224}wl^2, \quad M_5 = 0;$

$$R_1' = -\frac{11}{224}wl,\quad R_2'' = +\frac{11}{224}wl,\quad R_2' = +\frac{111}{224}wl,\quad R_3'' = +\frac{113}{224}wl,$$

$$R_3' = +\frac{15}{224}wl,\quad R_4'' = -\frac{15}{224}wl,\quad R_4' = -\frac{8}{224}wl,\quad R_5'' = +\frac{8}{224}wl.$$

(11) *Find the moment and reaction at the second support for a load W at a distance al from the left end of the second span.*

Ans. $M_2 = \frac{1}{56}(26a - 45a^2 + 19a^3)Wl$; $R_2' = \frac{W}{56}(56 - 38a - 57a^2 + 39a^3)$.

(12) *Deduce a formula for the moment at the left of any support of a continuous beam, level supports, when the entire beam is covered with the uniform load w per unit of length.*

Ans. $$M_n = -\frac{w}{4}\left[b_n - \frac{c_n[(l^3_{s-1} + l^3_s)d_s + (l^3_{s-2} + l^3_{s-1})d_{s-1} + \dots (l_1^3 + l_2^3)d_2]}{D}\right],$$

where the numbers b are given as follows:

$$b_1 = 0,\quad b_2 = 0,\quad b_3 = -\frac{l_1^3 + l_2^3}{l_2},\quad b_4 = -\frac{l_2^3 + l_3^3}{l_3} - 2b_3\frac{l_2 + l_3}{l_3},$$

$$b_5 = -\frac{l_3^3 + l_4^3}{l_4} - 2b_4\frac{l_3 + l_4}{l_4} - b_3\frac{l_3}{l_4},$$ etc., and in general

$$b_n = -\frac{l^{n-2}_{n-2} + l^{n-2}_{n-1}}{l_{n-1}} - 2b_{n-1}\frac{l_{n-2} + l_{n-1}}{l_{n-1}} - b_{n-2}\frac{l_{n-2}}{l_{n-1}}.$$

(13) *In the preceding example let the spans be all equal.*

Ans. $$M_n = -\frac{wl^2}{12c_{s+1}}[c_n(1 - c_{s+2}) - c_{s+1}(1 - c_{n+1})].$$

The following Table gives the coefficients of $+ wl^2$ for any number of spans. The Roman numerals at the sides indicate the number of spans, and the numbers in the spaces of each horizontal line give the moments on the left of each support.

MOMENTS ON LEFT OF SUPPORTS — TOTAL UNIFORM LOAD — LEVEL SUPPORTS—ALL SPANS EQUAL. COEFFICIENTS OF $+ wl^2$ GIVEN IN TABLE.

Number of Span

I	0	0							
II	0	1/8	0						
III	0	1/10	1/10	0					
IV	0	3/28	2/28	3/28	0				
V	0	4/38	3/38	3/38	4/38	0			
VI	0	11/104	8/104	9/104	8/104	11/104	0		
VII	0	15/142	11/142	12/142	12/142	11/142	15/142	0	
VIII	0	41/388	30/388	33/388	32/388	33/388	30/388	41/388	0

This Table may easily be continued to any number of spans. Thus for any *even* number of spans, as VIII for example, the coefficients are obtained by multiplying the fraction preceding in the same diagonal row, both numerator and denominator, by 2 and adding the numerator and denominator of the fraction preceding that. Thus,

$$\frac{15}{142}\frac{\times 2+}{\times 2+}\frac{11}{104}=\frac{41}{388},\qquad \frac{11}{142}\frac{\times 2+}{\times 2+}\frac{8}{104}=\frac{30}{388},$$

$$\frac{12}{142}\frac{\times 2+}{\times 2+}\frac{9}{104}=\frac{33}{388},\quad \text{or}\quad \frac{11}{142}\frac{\times 2+}{\times 2+}\frac{11}{104}=\frac{33}{388}.$$

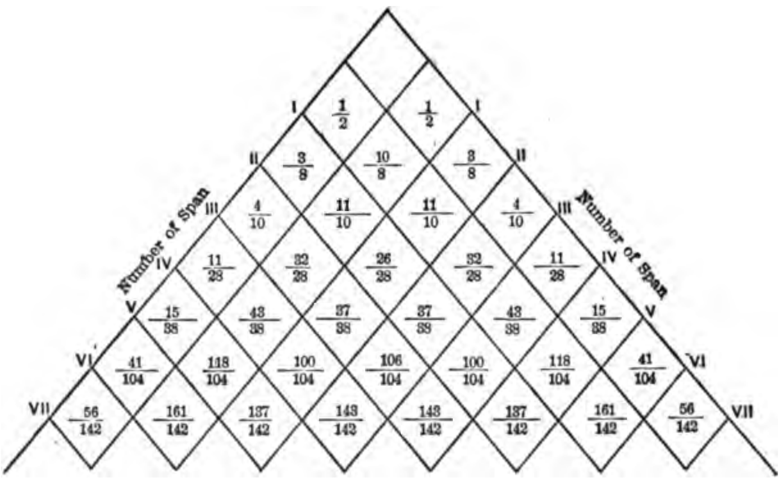
For any *odd* number of spans, as VII for example, we have simply to add, numerator to numerator and denominator to denominator, the two preceding fractions in the same diagonal row. Thus,

$$\frac{11}{104}\frac{+}{+}\frac{4}{38}=\frac{15}{142},\quad \frac{8}{104}\frac{+}{+}\frac{3}{38}=\frac{11}{142},\quad \frac{9}{104}\frac{+}{+}\frac{3}{38}=\frac{12}{142},\quad \text{or}\quad \frac{8}{104}\frac{+}{+}\frac{4}{38}=\frac{12}{142}.$$

The moments are all positive, showing that the upper fibre is in tension over every support.

The moments being known, the reactions can be found by (I), page 953. We then obtain the following Table.

REACTIONS AT SUPPORTS—TOTAL UNIFORM LOAD—LEVEL SUPPORTS—ALL SPANS EQUAL. COEFFICIENTS OF $+wl$ GIVEN IN TABLE.



The law of this Table is the same as for the preceding Table, and it can therefore be continued to any number of spans.

(14) *Give the formula for the moment at the left of any support of a continuous beam, level supports, for load in any given span only.*

Ans. From (III), page 375, let r be the left support of the loaded span. Then

$$M_n=\frac{d_{s-n}+{}_2A_rc_r+c_nd_{s-r}+{}_1B_r}{D}.$$

If the spans are all equal,

$$M_n = \frac{c_{s-n+2}A_r c_r + c_{s-r+1}B_r c_n}{D}.$$

If the spans are all equal and the span l_r is uniformly loaded with the load w per unit of length,

$$M_n = \frac{1}{4}wl^2\left[\frac{c_{s-n+2}c_r + c_{s-r+1}c_n}{D}\right].$$

(15) *A continuous beam of four equal spans, level supports, has the second span from the left covered with a uniform load of w per unit of length. Find the moments on the left of the supports and the reactions.*

$$\text{Ans. } M_1 = 0,\ \ M_2 = +\frac{11}{224}wl^2,\ \ M_3 = +\frac{12}{224}wl^2,\ \ M_4 = -\frac{3}{224}wl^2,\ \ M_5 = 0;$$

$$R_1' = -\frac{11}{224}wl,\ \ R_2'' = +\frac{11}{224}wl,\ \ R_2' = +\frac{111}{224}wl,\ \ R_3'' = +\frac{113}{224}wl,$$

$$R_3' = +\frac{15}{224}wl,\ \ R_4'' = -\frac{15}{224}wl,\ \ R_4' = -\frac{3}{224}wl,\ \ R_5'' = +\frac{3}{224}wl.$$

(16) *In the preceding case, what is the moment on left and reaction on right of the second support for a concentrated load W placed at a distance al from the left end of the second span?*

$$\text{Ans. } M_2 = \frac{1}{56}(26a - 45a^2 + 19a^3)Wl;$$

$$R_2' = \frac{W}{56}(56 - 38a - 57a^2 + 39a^3).$$

(17) *A continuous beam of five spans, the centre and adjacent spans being* 100 *feet and the end spans each* 75 *feet long, has a uniform load over the second span. Find the moments on the left of the supports, and the reaction on the right of the fourth support.*

$$\text{Ans. } M_1 = 0,\ \ M_2 = +\frac{35.5}{627}wl_2^2,\ \ M_3 = +\frac{65}{1254}wl_2^2,\ \ M_4 = -\frac{35}{2508}wl_2^2,$$

$$M_5 = +\frac{5}{1254}wl_2^2,\ M_6 = 0;\ \ R_4' = -\frac{45}{2508}wl_2.$$

(18) *A continuous beam of four spans,* $l_1 = 80$, $l_2 = 100$, $l_3 = 50$, $l_4 =$ 40 *feet, supports level, has a load of* 10 *tons in the second span, at a distance of* 40 *feet from the left end. Find the moments on left of the supports, and the reaction on the right of the second support.*

$$\text{Ans. } M_1 = 0,\ \ M_2 = \frac{Wl_2^2}{3348}(17a - 30.9a^2 + 13.9a^3) = +82.01 \text{ ft.-tons},$$

$$M_3 = \frac{3.6\,Wl_2^2}{3348}(1.6a + 3a^2 - 4.6a^3) = +88.77 \text{ ft.-tens},$$

$$M_4 = -\frac{Wl_2^2}{3348}(1.6a + 3a^2 - 4.6a^3) = -24.65 \text{ ft.-tons},\ \ M_5 = 0,$$

$$R_2' = +5.9324 \text{ tons}.$$

(16) *A beam continuous over seven spans has a load in every span. Find the moment on the left and reaction on right of the fourth support.*

Ans.

$$M_4 = -\frac{d_5}{d_8 l_1}[(Y_2 + A_2 + B_1)c_2 + (Y_3 + A_3 + B_2)c_3 + (Y_4 + A_4 + B_3)c_4]$$

$$- \frac{c_4}{d_8 l_1}[(Y_5 + A_5 + B_4)d_4 + (Y_6 + A_6 + B_5)d_3 + (Y_7 + A_7 + B_6)d_2],$$

$$M_5 = - \frac{d_4}{d_8 l_1}[(Y_2+A_2+B_1)c_2+(Y_3+A_3+B_2)c_3+(Y_4+A_4+B_3)c_4+(Y_5+A_5+B_4)c_5];$$

$$- \frac{c_5}{d_8 l_1}[(Y_6 + A_6 + B_5)d_3 + (Y_7 + A_7 + B_6)d_2];$$

$$R_4' = \frac{M_4 - M_5}{l_4} + q_4'.$$

(17) *Let the supports in* (16) *be on level, all spans equal,* $l = 80$ *feet, and only the first, third and sixth spans loaded with a uniform load* $w = 2$ *tons per unit of length.*

Ans. $M_4 = +\ 788.18$ ft.-tons, $M_5 = -\ 382.55$ ft.-tons;

$R_4' = +\ 14.63$ tons.

(18) *Let the supports in* (16) *be on level, all spans equal,* $l = 80$ *feet, and only the second, fifth and seventh spans loaded with a uniform load* $w = 2$ *tons per unit of length.*

Ans. $M_4 = -\ 382.55$ ft.-tons, $M_5 = +\ 788.18$ ft.-tons;

$R_4' = -\ 14.63$ tons.

(19) *Let the supports in* (16) *be on level, all spans equal,* $l = 80$ *feet, and a load* W *in the fourth span only at a distance* al *from the left end.*

Ans. $M_4 = \dfrac{15\,Wl}{2911}(97a - 168a^2 + 71a^3)$, $M_5 = \dfrac{15\,Wl}{2911}(26a + 45a^2 - 71a^3)$;

$$R_4' = \frac{15\,W}{2911}(71a - 213a^2 + 142a^3) + W(1 - a).$$

(20) *In* (19) *let a uniform load* w *per unit of length extend over the whole beam.*

Ans. $M_4 = +\dfrac{12}{142}wl^2$, $M_5 = +\dfrac{12}{142}wl^2$; $R_4' = +\dfrac{wl}{2}$.

(21) *Let the load in* (20) *be* 4000 *lbs. per ft. over the whole girder. How far must the fourth support be lowered in order that the moment at the fourth support may be zero?*

Ans. $h_3 - h_4 = -\dfrac{41wl^4}{1395EI}$

If $E = 24000000$ lbs. per square inch and $I = 53400$ for dimensions in inches, $h_3 - h_4 = -\ 6.5$ inches.

END OF VOL. II.

MECHANICS.

INDEX.

VOL. II.

STATICS.

A TEXT-BOOK ON THE MECHANICS OF MATERIALS,

And on Beams, Columns, and Shafts. By Prof. M. Merriman. Fourth edition, revised and enlarged..............8vo, cloth, $3 50

"We cannot commend this book too highly."—*American Engineer.*
"The well-earned reputation of the Author renders any comment on the quality of the work superfluous."—*Van Nostrand's Magazine.*

MECHANICS OF ENGINEERING.

Comprising Statics and Dynamics of Solids, the Mechanics of the Materials of Construction or Strength and Elasticity of Beams, Columns, Shafts, Arches, etc., and the Principles of Hydraulics and Pneumatics with Applications. For the use of Technical Schools. By Prof. Irving P. Church, C.E., Cornell University ....8vo, cloth, 6 00

"The work is very abundantly illustrated, and the information is given in a style which cannot fail to make the student thoroughly master of the subject. Prof. Church may certainly be congratulated upon compressing a vast amount of instruction into a very small space without in any degree interfering with the necessary minuteness of detail or clearness of description."—*London Industrial Review.*

MECHANICAL PRINCIPLES OF ENGINEERING AND ARCHITECTURE.

By Henry Mosely, M.A., F.R.S. From last London edition, with considerable additions by Prof. D. H. Mahan, LL.D., of the U. S. Military Academy. 700 pages. With numerous cuts...8vo, cloth, 5 00

MECHANICS OF ENGINEERING AND MACHINERY.

By Dr. Julius Weisbach. Designed as a Text-book for Technical Schools and Colleges, and for the use of Engineers, Draughtsmen, etc. Second edition, thoroughly revised and greatly enlarged, by Gustav Herrmann, Prof. at the Royal Polytechnic School, Aachen, Germany. Translated by J. F. Klein, D.E., Prof. of Mechanical Engineering, Lehigh University, Pa. With numerous fine illustrations. Second edition. 1 vol., 8vo, cloth, 5 00

"Weisbach is a standard in all matters of Engineering and Mechanics, and his teachings are accepted as correct."—*Mechanical Engineer.*

MECHANICS OF THE MACHINERY OF TRANSMISSION.

Being Vol. III., Part I., Section II. of Mechanics of Engineering and Machinery. By Dr. Julius Weisbach. Edited by Prof. Gustav Herrmann and translated by Prof. J. F. Klein, Lehigh University, Bethlehem, Pa......... ....8vo, cloth, 5 00

NOTES AND EXAMPLES IN MECHANICS.

With an Appendix on the Graphical Statics of Mechanism. By Prof. I. P. Church, Cornell University. 135 pages, with blank pages for problems........................8vo, cloth, 2 00

APPLIED MECHANICS AND RESISTANCE OF MATERIALS.

By Prof. G. Lanza. Showing Strains on Beams as determined by the Testing Machines of Watertown Arsenal and at the Massachusetts Institute of Technology. Practical and Theoretical. Designed for Engineers, Architects, and Students. With hundreds of illustrations. Sixth edition, revised.
1 vol., 8vo, cloth, $7 50

"The whole work is a valuable contribution to the subject of which it treats, and we can cordially recommend it."—*London Builder.*

WEISBACH'S MECHANICS. HYDRAULICS AND HYDRAULIC MOTORS.

With numerous practical examples for the calculation and construction of Water-wheels, including Breast, Undershot, Back-pitch, Overshot Wheels, etc., as well as a special discussion of the various forms of Turbines. Translated from the fourth edition of Weisbach's Mechanics, by A. Jay Du Bois. Profusely illustrated. Second edition.............8vo, cloth, 5 00

WEISBACH'S MECHANICS. THEORY OF STEAM-ENGINE.

Translated from the fourth edition of Weisbach's Mechanics by A. Jay Du Bois. Containing notes giving practical examples of Stationary, Marine, and Locomotive Engines, showing American practice, by R. H. Buel. Numerous illustrations. Second edition..8vo, cloth, 5 00

MECHANICS OF THE GIRDER.

A Treatise on Bridges and Roofs, in which the necessary and sufficient weight of the structure is calculated, not assumed, and the number of Panels and height of Girder that render the Bridge weight least for a given Span, Live Load, and Wind Pressure are determined. By John D. Crehore, C.E. Illustrated by over 100 engravings, with tables, etc. .... 8vo, cloth, 5 00

"The work bears internal evidences of patient industry and scholarly ability—is a valuable contribution to science and to the literature of Bridge building."—W. H. SEARLES, C.E.

KINEMATICS; OR, PRACTICAL MECHANISM.

A Treatise on the Transmission and Modification of Motion and the Construction of Mechanical Movements. For the use of Draughtsmen, Machinists, and Students of Mechanical Engineering, in which the laws governing the motions and various parts of Mechanics, as affected by their forms and modes of connection are deduced by simple geometrical reasoning, and their application is illustrated by accurately constructed diagrams of the different mechanical combinations discussed. By Prof. Chas. W. MacCord. Fourth edition......8vo, cloth, 5 00

"The work can be confidently recommended to Draughtsmen, and all who have occasion to design machinery, as well as to every earnest student of Mechanics, young or old."—*American Machinist.*

TREATISE ON FRICTION AND LOST WORK IN MACHINERY AND MILL WORK.

Containing an explanation of the Theory of Friction, and an account of the various Lubricants in general use, with a record of various experimenters to deduce the laws of Friction and Lubricated Surfaces, etc. By Prof. Robt. H. Thurston. Copiously illustrated. Fourth edition.............8vo, cloth, $3 00

"It is not too high praise to say that the present treatise is exhaustive and a complete review of the whole subject."—*American Engineer.*

CAR LUBRICATION.

Treating of Theoretical Relations, Coefficient of Friction, Bearing Metals, Methods of Lubrication, Journal-box Construction, Heated Journals, and the Cost of Lubrication. By W. E. Hall.............................12mo, cloth, 1 00

"A very useful book on a subject upon which literature is very scarce. While the author gives full credit to Prof. Thurston and to Mr. Woodbury for their researches in this direction, he puts the various theories and results of experiment in a very practical shape and shorn of all but the plainest mathematical dress. The volume is evidently the work of a practical investigator, and is correspondingly valuable."—*Engineering News.*

A HISTORY OF THE PLANING MILL.

With Practical Suggestions for the Construction, Care, and Management of Wood-working Machinery. By C. R. Tompkins, M.E..............................12mo, cloth, 1 50

"Each of these chapters is as full of meat as an egg; they give the results of long experience and intelligent observation, and no proprietor of woodworking machinery and employer of labor can afford to be without a copy, nor should any young mechanic, ambitious to excel in his calling, fail to send for it."—*The Lumberman*, Chicago, June 15, 1889.

DYNAMOMETERS, AND THE MEASUREMENT OF POWER.

By J. J. Flather, Prof. of Mechanical Engineering in Purdue University, Lafayette, Ind.....................12mo, cloth, 2 00

A Treatise on the Construction and Application of Dynamometers. Comprising Determination of Driving Power, Friction Brakes, Absorption and Transmission Dynamometers, Power to Drive Lathes, Measurement of Water-Power.

ELEMENTS OF MACHINE CONSTRUCTION AND DRAWING;

Or, Machine-Drawing, with some elements of descriptive and rational Kinematics. A Text-book for Schools of Civil and Mechanical Engineering, and for the use of Mechanical Establishments, Artisans, and Inventors. Containing the principles of Gearing, Screw Propellers, Valve Motions, and Governors, and many standard and novel examples, mostly from present American practice. By Prof. S. Edward Warren. Seventh edition..........2 vols., 8vo, text, and small 4to plates, cloth, 7 50

EXTRACTS FROM CHORDAL'S LETTERS.

Comprising the choicest selections from the Series of Articles which have been appearing for the past two years in the columns of the American Machinist. With over 50 illustrations. 12mo, cloth, 2 00

"The author discusses shop work and shop management with more practical shrewdness, and in a manner that Mechanics, Artisans, and wide awake working men, generally, cannot help but enjoy."—*Scientific American.*

THE LATHE AND ITS USES;

Or, Instruction in the Art of Turning Wood and Metal. Including a description of the most modern appliances for the ornamentation of plane and curved surfaces, with a description also of an entirely novel form of *Lathe* for Eccentric and Rose Engine Turning, a Lathe and Turning Machine combined, and other valuable matter relating to the Art. 1 vol., 8vo, copiously illustrated. Sixth edition, with additional chapters and Index.............................8vo, cloth, $6 00

"The most complete work on the subject ever published."—*American Artisan.*

"Here is an invaluable book to the practical workman and amateur."—*London Weekly Times.*

A TREATISE ON TOOTHED GEARING.

Containing complete instructions of Designing, Drawing, and Constructing Spur Wheels, Bevel Wheels, Lantern Gear, Screw Gear, Worms, etc., and the proper formation of Tooth Profiles. For the use of Machinists, Pattern Makers, Draughtsmen, Designers, Scientific Schools, etc. With many plates. By J. Howard Cromwell. Fourth edition............12mo, cloth, 1 50

"Mr. Cromwell has accomplished good work in bringing together in this volume a great deal of information only to be found by searching many works, and by adding the results of his own experience in the field of Mechanical Engineering."—*American Machinist.*

A TREATISE ON BELTS AND PULLEYS.

Embracing full explanations of Fundamental Principles; proper Disposition of Pulleys; Rules for determining widths of leather and vulcanized rubber belts, and belts running over covered pulleys; Strength and Proportions of Pulleys, Drums, etc. Together with the principles and necessary rules for Rope Gearing and transmission of power by means of Metallic Cables. By J. Howard Cromwell, Ph.B., author of a Treatise on Toothed Gearing..........................12mo, cloth, 1 50

"This is a very complete and comprehensive treatise, and is worthy of the attention of all Mechanics who have anything to do with the management of belts and pulleys, etc."—*National Car Builder.*

SAW FILING.

The Art of Saw Filing Scientifically Treated and Explained on Philosophical Principles. With explicit directions for putting in order all kinds of Saws, from a Jeweler's Saw to a Steam Saw-mill. Illustrated by 44 engravings. By H. W. Holly. Fifth edition...............................18mo, cloth, 0 75

SAW FILING.

A Practical Treatise on Filing, Gumming, and Swageing Saws. By Robert H. Grimshaw Fully illustrated......1 vol., 16mo, 1 00

MACHINERY PATTERN MAKING.

A Discussion of Methods, including Marking and Recording Patterns, Printing Press, Slice Valve and Corliss Cylinders; How to Cast Journal-boxes on Frames, Differential Pulleys, Fly-wheels, Engine Frames, Spur, Bevel and Worm Gears, Key Heads for Motion Rods, Elbows and Tee Pipes, Sweeping Straight and Conical Grooved Winding Drums, Large Sheaves with Wrought and Cast-iron Arms, 128 full-size Profiles of

Gear Teeth of different pitches for Gears of 14 to 800 Teeth, with a Table showing at a glance the required diameter of Gear for a given number of teeth and pitch, Double Beat, Governor, and Plug Valves, Screw Propellor, a chapter on items for Pattern Makers, besides a number of valuable and useful Tables, etc., etc. 417 illustrations By P S. Dingey, Foreman Pattern Maker and Draughtsman. ...12mo, cloth, $2 00

"A neat little work that should be not only in the hands of every pattern maker, but read by every foundry foreman and proprietor of foundries doing machinery work."—*Machinery Moulder's Journal.*

THE BOSTON MACHINIST.

Being a complete School for the Apprentice as well as the advanced Machinist, showing how to make and use every tool in every branch of the business; with a Treatise on Screws and Gear-cutting. By Walter Fitzgerald. Third ed'n. 18mo, cloth, 0 75

STEAM HEATING FOR BUILDINGS.

Or, Hints to Steam Fitters. Being a description of Steam Heating Apparatus for Warming and Ventilating Private Houses and Large Buildings, with Remarks on Steam, Water, and Air in their Relations to Heating. To which are added useful miscellaneous tables. By Wm. J. Baldwin. Thirteenth edition. With many illustrative plates..........12mo, cloth, 2 50

"Mr. Baldwin has supplied a want long felt for a practical work on Heating and Heating Apparatus."—*Sanitary Engineer.*

THE COST OF MANUFACTURES—AND THE ADMINISTRATION OF WORKSHOPS, PUBLIC AND PRIVATE.

A System of Mechanical Book-keeping, based on the Card-Catalogue method, dispensing with skilled clerical labor and the use of books, by which the cost of manufactures may be promptly determined, either in gross or in any detail, as to component parts and operations thereon. Comprising a simple method of recording all dealings with materials which relate to its procurement, expenditure, or possession. Applied, with numerous practical illustrations, to the trust, accountability for public property, and funds required of the U. S. Ordnance Department, with a review of its present practice. By Capt. Henry Metcalfe, U. S. Ordnance Department. Illustrated with tables, forms of cards, etc., etc. Second ed'n. 8vo, cloth, 5 00

"I feel sure that by the use of your methods I can determine a cost I have never been able to arrive at."—Ewart Manufacturing Co., *Chicago.*

"We find that it enables us to keep a more accurate record of each piece of work. We can locate the responsibility for any delay or omission."—Rathbone & Co., Stove Works, Albany.

WRINKLES AND RECIPES.

Compiled from the SCIENTIFIC AMERICAN. A collection of practical suggestions, processes, and directions, for the MECHANIC, ENGINEER, FARMER, and Housekeeper. With a COLOR TEMPERING SCALE, and numerous wood engravings. By Park Benjamin. Revised by Profs. Thurston and Van der Weyde, and Engineers Buel and Rose. Fifth revised edition.... ....12mo, cloth, 2 00

"Hundreds of Trade Secrets and Mechanical Shop Wrinkles."